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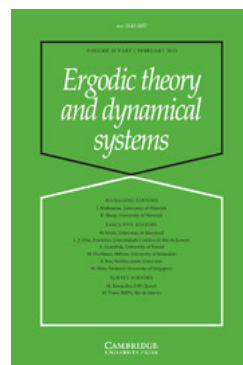
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Weak equivalence and non-classifiability of measure preserving actions

ROBIN D. TUCKER-DROB

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Weak equivalence and non-classifiability of measure preserving actions

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Abstract. Abért and Weiss have shown that the Bernoulli shift s_Γ of a countably infinite group Γ is weakly contained in any free measure preserving action α of Γ . Proving a conjecture of Ioana, we establish a strong version of this result by showing that $s_\Gamma \times \alpha$ is weakly equivalent to α . Using random Bernoulli shifts introduced by Abért, Glasner, and Virag, we generalize this to non-free actions, replacing s_Γ with a random Bernoulli shift associated to an invariant random subgroup, and replacing the product action with a relatively independent joining. The result for free actions is used along with the theory of Borel reducibility and Hjorth’s theory of turbulence to show that, on the weak equivalence class of a free measure preserving action, the equivalence relations of isomorphism, weak isomorphism, and unitary equivalence are not classifiable by countable structures. This in particular shows that there are no free *weakly rigid* actions, that is, actions whose weak equivalence class and isomorphism class coincide, answering negatively a question of Abért and Elek. We also answer a question of Kechris regarding two ergodic theoretic properties of residually finite groups. A countably infinite residually finite group Γ is said to have property EMD* if the action p_Γ of Γ on its profinite completion weakly contains all ergodic measure preserving actions of Γ , and Γ is said to have property MD if $\iota \times p_\Gamma$ weakly contains all measure preserving actions of Γ , where ι denotes the identity action on a standard non-atomic probability space. Kechris has shown that EMD* implies MD and asked if the two properties are actually equivalent. We provide a positive answer to this question by studying the relationship between convexity and weak containment in the space of measure preserving actions.

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1. Introduction

By a *measure preserving action* of a countable group Γ we mean a triple $(\Gamma, a, (X, \mu))$, which we write as $\Gamma \curvearrowright^a (X, \mu)$, where (X, μ) is a standard probability space (that is, a standard Borel space equipped with a Borel probability measure) and $a : \Gamma \times X \rightarrow X$ is a Borel action of Γ on X that preserves the Borel probability measure μ . In what follows all measures are probability measures unless explicitly stated otherwise and we will write \mathbf{a} and \mathbf{b} to denote the measure preserving actions $\Gamma \curvearrowright^a (X, \mu)$ and $\Gamma \curvearrowright^b (Y, \nu)$ respectively when the group Γ and the underlying probability spaces (X, μ) and (Y, ν) are understood. Given measure preserving actions $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ and $\mathbf{b} = \Gamma \curvearrowright^b (Y, \nu)$, we say that \mathbf{a} is *weakly contained* in \mathbf{b} , and write $\mathbf{a} \prec \mathbf{b}$, if for every finite partition A_0, \dots, A_{k-1} of X into Borel sets, every finite subset $F \subseteq \Gamma$, and every $\epsilon > 0$, there exists a Borel partition B_0, \dots, B_{k-1} of Y such that

$$|\mu(\gamma^a A_i \cap A_j) - \nu(\gamma^b B_i \cap B_j)| < \epsilon$$

for all $\gamma \in F$ and $0 \leq i, j < k$. We write $\mathbf{a} \sim \mathbf{b}$ if both $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{b} \prec \mathbf{a}$, in which case \mathbf{a} and \mathbf{b} are said to be *weakly equivalent*. The notion of weak containment of measure preserving actions was introduced by Kechris [Ke10] as an ergodic theoretic analogue of weak containment for unitary representations.

Weak containment of unitary representations may be defined as follows (see [BdlHV08, Appendix F]). Let π and ρ be unitary representations of Γ on the Hilbert spaces \mathcal{H}_π and \mathcal{H}_ρ respectively. Then π is *weakly contained* in ρ , written $\pi < \rho$, if for every unit vector ξ in \mathcal{H}_π , every finite subset $F \subseteq \Gamma$, and every $\epsilon > 0$, there exists a finite collection $\eta_0, \dots, \eta_{k-1}$ of unit vectors in \mathcal{H}_ρ and non-negative real numbers $\alpha_0, \dots, \alpha_{k-1}$ with $\sum_{i=0}^{k-1} \alpha_i = 1$ such that

$$\left| \langle \pi(\gamma)\xi, \xi \rangle - \sum_{i=0}^{k-1} \alpha_i \langle \rho(\gamma)\eta_i, \eta_i \rangle \right| < \epsilon$$

for all $\gamma \in F$. Each unit vector $\xi \in H_\pi$ gives rise to a normalized positive definite function on Γ defined by $\gamma \mapsto \langle \pi(\gamma)\xi, \xi \rangle$. We call such a function a *normalized positive definite function realized in π* and we may rephrase the definition of $\pi < \rho$ accordingly as: every normalized positive definite function realized in π is a pointwise limit of convex sums of normalized positive definite functions realized in ρ .

A similar rephrasing also applies to weak containment of measure preserving actions, as pointed out by Abért and Weiss in [AW13]. If we view a finite Borel partition A_0, \dots, A_{k-1} of X as a Borel function $\phi : X \rightarrow k = \{0, 1, \dots, k-1\}$ (where we view k as a discrete space) then, given a measure preserving action $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$, each partition $\phi : X \rightarrow k$ gives rise to a shift-invariant Borel probability measure $(\Phi^{\phi, \mathbf{a}})_* \mu$ on k^Γ , where

$$\Phi^{\phi, \mathbf{a}}(x)(\gamma) = \phi((\gamma^{-1})^a \cdot x).$$

The map $\Phi^{\phi, \mathbf{a}}$ is equivariant between the action \mathbf{a} and the shift action s on k^Γ given by $(\gamma^s \cdot f)(\delta) = f(\gamma^{-1}\delta)$, and one may show that the measures $(\Phi^{\phi, \mathbf{a}})_* \mu$, as ϕ ranges over all Borel partitions of X into k -pieces, are precisely those shift-invariant Borel measures λ such that $\Gamma \curvearrowright^s (k^\Gamma, \lambda)$ is a factor of \mathbf{a} . In this language \mathbf{a} being weakly contained in \mathbf{b} means that for every natural number k , each shift-invariant measure on k^Γ that is a factor of \mathbf{a} is a weak* limit of shift-invariant measures that are factors of \mathbf{b} .

More precisely, given a compact Polish space K we equip K^Γ with the product topology, and we let $M_s(K^\Gamma)$ denote the convex set of shift-invariant Borel probability measures on K^Γ equipped with the weak* topology so that it is also a compact Polish space. We define

$$E(\mathbf{a}, K) = \{(\Phi^{\phi, \mathbf{a}})_* \mu : \phi : X \rightarrow K \text{ is Borel}\} \subseteq M_s(K^\Gamma).$$

Then the Abért–Weiss characterization of weak containment of measure preserving actions may be stated as follows: $\mathbf{a} < \mathbf{b}$ if and only if $E(\mathbf{a}, K) \subseteq \overline{E(\mathbf{b}, K)}$ for every finite K if and only if $E(\mathbf{a}, K) \subseteq \overline{E(\mathbf{b}, K)}$ for every compact Polish space K .

From this point of view one difference between the two notions of weak containment is apparent. While weak containment of representations allows for normalized positive definite functions realized in π to be approximated by *convex sums* of normalized positive definite functions realized in ρ , weak containment of measure preserving actions asks that shift invariant factors of \mathbf{a} be approximated by a *single* shift invariant factor of \mathbf{b} at a time. It is natural to ask for a characterization of the situation in which shift invariant factors of \mathbf{a} are approximated by convex sums of shift invariant factors of \mathbf{b} . When

this is the case we say that \mathbf{a} is *stably weakly contained in* \mathbf{b} and we write $\mathbf{a} \prec_s \mathbf{b}$. The relationship between weak containment and stable weak containment of measure preserving actions is analogous to the relationship between weak containment in the sense of Zimmer (see [BdlHV08, Remark F.1.2(ix)] and [Ke10, Appendix H (B)]) and weak containment of unitary representations. Our first theorem is a characterization of this stable version of weak containment of measure preserving actions.

In what follows (X, μ) and (Y, ν) and (Z, η) always denote standard probability spaces. We let $\iota_\eta : \Gamma \times Z \rightarrow Z$ denote the trivial (identity) action of Γ on (Z, η) , writing ι_η for the corresponding triple $\Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$, and we write ι and \mathbf{t} for ι_η and ι_η respectively when η is non-atomic. We show the following result in §3.

THEOREM 1.1. *Let $\mathbf{b} = \Gamma \curvearrowright^b (Y, \nu)$ be a measure preserving action of Γ . Then $\overline{E}(\mathbf{t} \times \mathbf{b}, K) = \overline{\text{co}}E(\mathbf{b}, K)$ for every compact Polish space K . In particular, for any $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ we have that $\mathbf{a} \prec \mathbf{t} \times \mathbf{b}$ if and only if $E(\mathbf{a}, K) \subseteq \overline{\text{co}}E(\mathbf{b}, K)$ for every compact Polish space K .*

When \mathbf{a} is ergodic, so that $E(\mathbf{a}, K)$ is contained in the extreme points of $M_s(K^\Gamma)$, we show that Theorem 1.1 implies the following direct analogue of the fact (see [BdlHV08, F.1.4]) that if π and ρ are representations of Γ , π is irreducible, and π is weakly contained in ρ , then every normalized positive definite function realized in π is actually a pointwise limit of normalized positive definite functions realized in ρ .

THEOREM 1.2. *Let $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ and $\mathbf{b} = \Gamma \curvearrowright^b (X, \mu)$ be measure preserving actions of Γ and suppose that \mathbf{a} is ergodic. If $\mathbf{a} \prec \mathbf{t} \times \mathbf{b}$ then $\mathbf{a} \prec \mathbf{b}$.*

In Theorem 3.12 we show more generally that if \mathbf{a} is an ergodic measure preserving action that is weakly contained in \mathbf{d} , then \mathbf{a} is weakly contained in almost every ergodic component of \mathbf{d} . This may be seen as a weak containment analogue of the fact that if \mathbf{a} is a factor of \mathbf{d} , then \mathbf{a} is a factor of almost every ergodic component of \mathbf{d} (see Proposition 3.9 below).

One consequence of Theorem 1.2 is that every non-amenable group has a free, non-ergodic weak equivalence class, and this in fact characterizes non-amenability (Corollary 4.2 below). Recall that an ergodic measure preserving action \mathbf{a} is *strongly ergodic* if and only if it does not weakly contain the identity action \mathbf{t} .

THEOREM 1.3. *If \mathbf{b} is a measure preserving action of Γ that is strongly ergodic then $\mathbf{t} \times \mathbf{b}$ is not weakly equivalent to any ergodic action. In particular, if Γ is a non-amenable group and $s_\Gamma = \Gamma \curvearrowright^{s_\Gamma} ([0, 1]^\Gamma, \lambda^\Gamma)$ is the Bernoulli shift action of Γ , then $\mathbf{t} \times s_\Gamma$ is a free action of Γ that is not weakly equivalent to any ergodic action.*

If \mathcal{B} is a class of measure preserving actions of a countable group Γ and $\mathbf{a} \in \mathcal{B}$, then \mathbf{a} is called *universal for* \mathcal{B} if $\mathbf{b} \prec \mathbf{a}$ for every $\mathbf{b} \in \mathcal{B}$. When \mathbf{a} is universal for the class of *all* measure preserving actions of Γ then \mathbf{a} is simply called *universal*. In §4 we study the universality properties EMD, EMD*, and MD of residually finite groups introduced by Kechris [Ke12] (MD was also independently studied by Bowen [Bo03], but with different terminology), and defined as follows. A countably infinite group Γ is said to have property EMD if the measure preserving action \mathbf{p}_Γ of Γ on its profinite completion is universal.

Γ is said to have property EMD* if p_Γ is universal for the class of all ergodic measure preserving actions of Γ . Γ is said to have property MD if $\iota \times p_\Gamma$ is universal.

Each of these properties imply that Γ is residually finite and it is clear that EMD implies both EMD* and MD. Kechris shows that EMD* implies MD and asks [Ke12, Question 4.11] whether the converse is true. We provide a positive answer to this question.

THEOREM 1.4. *The properties MD and EMD* are equivalent.*

This implies (Corollary 4.7 below) that the properties EMD and MD are equivalent for all groups without property (T). We also show in Theorem 4.8 that the free product of groups with property MD has EMD and we give two reformulations of the problem of whether EMD and MD are equivalent in general (Theorem 4.10 below).

In §5 we discuss the structure of weak equivalence with respect to invariant random subgroups. A countable group Γ acts on the compact space $\text{Sub}(\Gamma) \subseteq 2^\Gamma$ of all of its subgroups by conjugation. Following [AGV12], a conjugation-invariant Borel probability measure on $\text{Sub}(\Gamma)$ will be called an *invariant random subgroup* (IRS) of Γ . We let $\text{IRS}(\Gamma)$ denote the set of all invariant random subgroups of Γ . If $\mathbf{a} = \Gamma \curvearrowright^a (Y, \nu)$ is a measure preserving action of Γ then the stabilizer map $y \mapsto \text{stab}_a(y) \in \text{Sub}(\Gamma)$ is equivariant so that the measure $(\text{stab}_a)_* \nu$ is an IRS of Γ which we call the *type* of \mathbf{a} , and denote $\text{type}(\mathbf{a})$. It is shown in [AE11] that the type of a measure preserving action is an invariant of weak equivalence (we give a proof of this in Theorem 5.2 below).

In §5.2 we use the framework laid out in §3 to study the compact metric topology introduced by Abért and Elek [AE11] on the set $A_\sim(\Gamma, X, \mu)$ of weak equivalence classes of measure preserving actions of Γ . We show that the map $A_\sim(\Gamma, X, \mu) \rightarrow \text{IRS}(\Gamma)$ sending each weak equivalence class to its type in $\text{IRS}(\Gamma)$ is continuous when $\text{IRS}(\Gamma)$ is equipped with the weak* topology.

In §5.3 we detail a construction, described in [AGV12], whereby, given a probability space (Z, η) , one canonically associates to each $\theta \in \text{IRS}(\Gamma)$ a measure preserving action $s_{\theta, \eta}$ of Γ such that $\text{type}(s_{\theta, \eta}) = \theta$ when η is non-atomic. We call $s_{\theta, \eta}$ the θ -random Bernoulli shift of Γ over (Z, η) . When \mathbf{a} is free then $\text{type}(\mathbf{a})$ is the point mass $\delta_{\langle e \rangle}$ on the trivial subgroup $\langle e \rangle$ of Γ and $s_{\delta_{\langle e \rangle}, \eta}$ is the usual Bernoulli shift action of Γ on (Z^Γ, η^Γ) . After establishing some properties of random Bernoulli shifts we show the following in §5.5.

THEOREM 1.5. *Let $\mathbf{a} = \Gamma \curvearrowright^a (Y, \nu)$ be a non-atomic measure preserving action of type θ , and let $s_{\theta, \eta}$ be the θ -random Bernoulli shift over (Z, η) . Then the relatively independent joining of $s_{\theta, \eta}$ and \mathbf{a} over their common factor $\Gamma \curvearrowright (\text{Sub}(\Gamma), \theta)$ is weakly equivalent to \mathbf{a} . In particular, $s_{\theta, \eta}$ is weakly contained in every non-atomic action of type θ .*

When \mathbf{a} is free, the relatively independent joining of $s_{\delta_{\langle e \rangle}, \eta}$ and \mathbf{a} is simply the product of the Bernoulli shift with \mathbf{a} and Theorem 1.5 proves a conjecture of Ioana, becoming the following strengthening of [AW13, Theorem 1].

COROLLARY 1.6. *Let $s_\Gamma = \Gamma \curvearrowright^{s_\Gamma} ([0, 1]^\Gamma, \lambda^\Gamma)$ be the Bernoulli shift action of Γ , where λ denotes Lebesgue measure on $[0, 1]$. Let $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ be a free measure preserving action of Γ on a non-atomic standard probability space (X, μ) . Then $s_\Gamma \times \mathbf{a}$ is weakly equivalent to \mathbf{a} .*

Several invariants of measure preserving actions such as groupoid cost [AW13] ([Ke10] for the case of free actions) and independence number [CK13] are known to be monotonic with respect to weak containment (see also [AE11, CKT-D13] for other examples). A consequence of Theorem 1.5 is that, for a finitely generated group Γ , among all non-atomic measure preserving actions of type θ , the groupoid cost attains its maximum and the independence number attains its minimum on $s_{\theta, \lambda}$. Likewise, Corollary 1.6 implies that for any free measure preserving action \mathbf{a} of Γ , both \mathbf{a} and $s_\Gamma \times \mathbf{a}$ have the same independence number, and the orbit equivalence relation associated to \mathbf{a} and $s_\Gamma \times \mathbf{a}$ have the same cost.

In §6 we address the question of how many isomorphism classes of actions are contained in a given weak equivalence class. We answer a question of Abért and Elek [AE11, Question 6.1], showing that the weak equivalence class of any free action always contains non-isomorphic actions. Our arguments show that there are in fact continuum many isomorphism classes of actions in any free weak equivalence class, and from the perspective of Borel reducibility we can strengthen this even further. Let $A(\Gamma, X, \mu)$ denote the Polish space of measure preserving actions of Γ on (X, μ) and let $\mathbf{a}, \mathbf{b} \in A(\Gamma, X, \mu)$. Then \mathbf{a} and \mathbf{b} are called *weakly isomorphic*, written $\mathbf{a} \cong^w \mathbf{b}$, if both $\mathbf{a} \sqsubseteq \mathbf{b}$ and $\mathbf{b} \sqsubseteq \mathbf{a}$. We call \mathbf{a} and \mathbf{b} *unitarily equivalent*, written $\mathbf{a} \cong^u \mathbf{b}$, if the corresponding Koopman representations $\kappa_0^{\mathbf{a}}$ and $\kappa_0^{\mathbf{b}}$ are unitarily equivalent. We let \cong denote isomorphism of actions. Then $\mathbf{a} \cong \mathbf{b} \Rightarrow \mathbf{a} \cong^w \mathbf{b} \Rightarrow \mathbf{a} \cong^u \mathbf{b}$. We now have the following theorem.

THEOREM 1.7. *Let $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ be a free action of a countably infinite group Γ and let $[\mathbf{a}] = \{\mathbf{b} \in A(\Gamma, X, \mu) : \mathbf{b} \sim \mathbf{a}\}$ be the weak equivalence class of \mathbf{a} . Then isomorphism on $[\mathbf{a}]$ does not admit classification by countable structures. The same holds for both weak isomorphism and unitary equivalence on $[\mathbf{a}]$.*

Any two free actions of an infinite amenable group are weakly equivalent ([FW04], see also Remark 4.1 and Theorem 1.8 below), so for amenable Γ Theorem 1.7 follows from [FW04], [Hj97] and [Ke10, Theorems 13.7, 13.8, and 13.9] (see also [KLP10, 4.4]), while for non-amenable Γ there are continuum many weak equivalence classes of free actions (see Remark 4.3 below), and Theorem 1.7 is therefore a refinement of the existing results. The proof of Theorem 1.7 uses the methods of [Ke10, Theorem 13.7] and [KLP10]. We fix an infinite dimensional separable Hilbert space \mathcal{H} , and denote by $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ the Polish space of unitary representations of Γ on \mathcal{H} that are weakly contained in the left regular representation λ_Γ of Γ . The conjugacy action of the unitary group $\mathcal{U}(\mathcal{H})$ on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ is generically turbulent by [KLP10, 3.3], so Theorem 1.7 will follow by showing that unitary conjugacy on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ is not generically $\cong[\mathbf{a}]$ -ergodic (and that the same holds for \cong^w and \cong^u in place of \cong). For this we find a continuous homomorphism ψ from unitary conjugacy on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ to isomorphism on $[\mathbf{a}]$ with the property that the inverse image of each \cong^u -class is meager. The main new ingredient that is needed in the proof of Theorem 1.7 is Corollary 1.6, which shows that the homomorphism ψ we define takes values in $[\mathbf{a}]$.

In §7 we show that when Γ is amenable, $\text{type}(\mathbf{a})$ completely determines the stable weak equivalence class (Definition B.1) of a measure preserving action \mathbf{a} of Γ .

THEOREM 1.8. Let \mathbf{a} and \mathbf{b} be two measure preserving actions of an amenable group Γ .

- (1) Then $\text{type}(\mathbf{a}) = \text{type}(\mathbf{b})$ if and only if $\mathbf{a} \sim_s \mathbf{b}$.
- (2) Suppose that $\text{type}(\mathbf{a}) = \text{type}(\mathbf{b})$ concentrates on the infinite index subgroups of Γ . Then $\mathbf{a} \sim \mathbf{b}$.

Combining this with the results of §5.2 (see Remark 5.8) shows that when Γ is amenable, the type map $[\mathbf{a}]_s \mapsto \text{type}(\mathbf{a})$, from the compact space $A_{\sim_s}(\Gamma, Y, \nu)$ of all stable weak equivalence classes of measure preserving actions of Γ , to the space $\text{IRS}(\Gamma)$, is a homeomorphism.

We end with two appendices, one on ultraproducts of measure preserving actions, and one on stable weak containment.

Remark 1.9. After sending Gábor Elek a preliminary version of this paper, I was informed by him that he has independently obtained a version of Theorem 1.8. See [Eli12].

2. Preliminaries and notation

Γ will always denote a countable group, and e will always denote the identity element of Γ .

2.1. Measure algebras and standard probability spaces. All measures will be probability measures unless explicitly stated otherwise. A *standard probability space* is a probability measure space $(X, \mu) = (X, \mathbf{B}(X), \mu)$, where X is a standard Borel space and μ is a probability measure on the σ -algebra $\mathbf{B}(X)$ of Borel subsets of X . In what follows, (X, μ) , (Y, ν) , and (Z, η) will always denote standard probability spaces. Though we mainly focus on standard probability spaces we will make use of non-standard probability spaces arising as ultraproducts of standard probability spaces. We will write (W, ρ) for a probability space that may or may not be standard.

The *measure algebra* MALG_ρ of a probability space (W, ρ) is the σ -algebra of ρ -measurable sets modulo the σ -ideal of null sets, equipped with the measure ρ . We also equip MALG_ρ with the metric $d_\rho(A, B) = \rho(A \Delta B)$. We will sometimes abuse notation and identify a measurable set $A \subseteq W$ with its equivalence class in MALG_ρ when there is no danger of confusion.

2.2. Measure preserving actions. Let Γ be a countable group. A *measure preserving action* of Γ is a triple $(\Gamma, a, (X, \mu))$, which we write as $\Gamma \curvearrowright^a (X, \mu)$, where (X, μ) is a standard probability space and $a : \Gamma \times X \rightarrow X$ is a Borel action of Γ on X that preserves the probability measure μ . A measure preserving action $\Gamma \curvearrowright^a (X, \mu)$ will often also be denoted by a boldface letter such as \mathbf{a} or $\boldsymbol{\mu}$ depending on whether we want to emphasize the underlying action or the underlying probability measure. When $\gamma \in \Gamma$ and $x \in X$ we write $\gamma^a \cdot x$ or $\gamma^a x$ for $a(\gamma, x)$. In what follows, \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} will always denote measure preserving actions of Γ .

We will also make use of actions of Γ on non-standard probability spaces. When (W, ρ) is a probability space and $o : \Gamma \times W \rightarrow W$ is a measurable action of Γ on W that preserves ρ , we will still use the notations $\mathbf{o} = \Gamma \curvearrowright^o (W, \rho)$, γ^o , etc., from above, *though we reserve the phrase ‘measure preserving action’ for the case when the underlying probability space is standard.*

2.3. The space of measure preserving actions. We let $A(\Gamma, X, \mu)$ denote the set of all measure preserving actions of Γ on (X, μ) modulo almost everywhere equality. That is, two measure preserving actions \mathbf{a} and \mathbf{b} of Γ on (X, μ) are equivalent if $\mu(\{x \in X : \gamma^{\mathbf{a}}x \neq \gamma^{\mathbf{b}}x\}) = 0$ for all $\gamma \in \Gamma$. Though elements of $A(\Gamma, X, \mu)$ are equivalence classes of measure preserving actions we will abuse notation and confuse elements of $A(\Gamma, X, \mu)$ with their Borel representatives, making sure our statements and definitions are independent of the choice of representative when it is not obvious. We equip $A(\Gamma, X, \mu)$ with the *weak topology*, which is a Polish topology generated by the maps $\mathbf{a} \mapsto \gamma^{\mathbf{a}}A \in \text{MALG}_{\mu}$, with A ranging over MALG_{μ} and γ ranging over elements of Γ .

Notation. For $\mathbf{a} \in A(\Gamma, X, \mu)$ and $\mathbf{b} \in A(\Gamma, Y, \nu)$ we let $\mathbf{a} \sqsubseteq \mathbf{b}$ denote that \mathbf{a} is a factor of \mathbf{b} and we let $\mathbf{a} \cong \mathbf{b}$ denote that \mathbf{a} and \mathbf{b} are isomorphic. We let $\iota_{\eta} \in A(\Gamma, Z, \eta)$ denote the trivial (identity) system $\Gamma \curvearrowright^{\iota_{\eta}} (Z, \eta)$, and we write ι for ι_{η} when η is non-atomic. We call $\Gamma \curvearrowright^{\mathbf{a}} (X, \mu)$ *non-atomic* if the probability space (X, μ) is non-atomic. If $T : X \rightarrow X$ then we let $\text{supp}(T) = \{x \in X : T(x) \neq x\}$. For a $A \subseteq X$ we denote by $\mu|A$ the restriction of μ to A given by $(\mu|A)(B) = \mu(B \cap A)$ and we denote by μ_A the conditional probability measure $\mu_A(B) = \mu(B \cap A)/\mu(A)$, where we use the convention that $\mu_A \equiv 0$ when $A \subseteq X$ is null.

Convention. We will regularly neglect null sets when there is no danger of confusion.

3. Proofs of Theorems 1.1 and 1.2

3.1. Weak containment and shift-invariant factors. Let K be a compact Polish space and equip K^{Γ} with the product topology so that it is also a compact Polish space. Then Γ acts continuously on K^{Γ} by the shift action s , given by $(\delta^s f)(\gamma) = f(\delta^{-1}\gamma)$ for $\delta, \gamma \in \Gamma$, $f \in K^{\Gamma}$. Let (W, ρ) be a probability space and let $\mathbf{o} = \Gamma \curvearrowright^{\mathbf{o}} (W, \rho)$ be a measurable action of Γ on W that preserves ρ . For each measurable function $\phi : W \rightarrow K$ we define $\Phi^{\phi, \mathbf{o}} : W \rightarrow K^{\Gamma}$ by $\Phi^{\phi, \mathbf{o}}(w)(\gamma) = \phi((\gamma^{-1})^{\mathbf{o}} \cdot w)$, and we let

$$E(\mathbf{o}, K) = \{(\Phi^{\phi, \mathbf{o}})_{*}\rho : \phi : W \rightarrow K \text{ is } \rho\text{-measurable}\}.$$

Each map $\Phi^{\phi, \mathbf{o}}$ is a factor map from \mathbf{o} to $\Gamma \curvearrowright^s (K^{\Gamma}, (\Phi^{\phi, \mathbf{o}})_{*}\rho)$ since

$$\begin{aligned} \Phi^{\phi, \mathbf{o}}(\delta^{\mathbf{o}} \cdot w)(\gamma) &= \phi((\gamma^{-1}\delta)^{\mathbf{o}} \cdot w) = \phi(((\delta^{-1}\gamma)^{-1})^{\mathbf{o}} \cdot w) \\ &= \Phi^{\phi, \mathbf{o}}(w)(\delta^{-1}\gamma) = (\delta^s \cdot \Phi^{\phi, \mathbf{o}}(w))(\gamma). \end{aligned}$$

Conversely, given any measurable factor map $\psi : \Gamma \curvearrowright^{\mathbf{o}} (W, \rho) \rightarrow \Gamma \curvearrowright^s (K^{\Gamma}, \pi_{*}\mu)$ the map $\phi(w) = \psi(w)(e)$ is also measurable, and for almost all $w \in W$ and all $\gamma \in \Gamma$ we have

$$\Phi^{\phi, \mathbf{o}}(w)(\gamma^{-1}) = \phi(\gamma^{\mathbf{o}} \cdot w) = \psi(\gamma^{\mathbf{o}} \cdot w)(e) = (\gamma^s \cdot \psi(w))(e) = \psi(w)(\gamma^{-1})$$

so that $\psi_{*}\rho = (\Phi^{\phi, \mathbf{o}})_{*}\rho$. It follows that $E(\mathbf{o}, K)$ is the set of all shift-invariant Borel probability measures on K^{Γ} that are factors of \mathbf{o} . We let $M_s(K^{\Gamma})$ denote the convex set of all shift-invariant Borel probability measures on K^{Γ} . Equipped with the weak* topology, this is a compact metrizable subset of $C(K^{\Gamma})^*$.

Definition 3.1. If $E \subseteq M_s(K^{\Gamma})$ we let $\text{co}E$ denote the convex hull of E and we let $\overline{\text{co}}E$ denote the closed convex hull of E . We write $\text{ex}(E)$ for the set of extreme points of a convex set $E \subseteq M_s(K^{\Gamma})$.

LEMMA 3.2. Suppose that $\phi_n : W \rightarrow K$, $n \in \mathbb{N}$, is a sequence of measurable functions that converge in measure to the measurable function $\phi : W \rightarrow K$. Then $(\Phi^{\phi_n, o})_* \rho \rightarrow (\Phi^{\phi, o})_* \rho$ in $M_s(K^\Gamma)$.

Proof. ϕ_n converges to ϕ in measure if and only if for every subsequence $\{n_i\}$ there is a further subsequence $\{m_i\}$ such that $\phi_{m_i} \rightarrow \phi$ almost surely. If $\phi_{m_i} \rightarrow \phi$ almost surely then for all $\gamma \in \Gamma$, $\Phi^{\phi_{m_i}, o}(w)(\gamma) \rightarrow \Phi^{\phi, o}(w)(\gamma)$ almost surely, and so $\Phi^{\phi_{m_i}, o}(w) \rightarrow \Phi^{\phi, o}(w)$ almost surely. It follows that $\Phi^{\phi_n, o} \rightarrow \Phi^{\phi, o}$ in measure. Since convergence in measure implies convergence in distribution it follows that $(\Phi^{\phi_n, o})_* \rho \rightarrow (\Phi^{\phi, o})_* \rho$ in $M_s(K^\Gamma)$. \square

Remark 3.3. We may form the space $L(W, \rho, K)$ of all measurable maps $\phi : W \rightarrow K$, where we identify two such maps if they agree ρ -almost everywhere. If $d \leq 1$ is a compatible metric for K then we equip $L(W, \rho, K)$ with the metric $\tilde{d}(\phi, \psi) = \int_W d(\phi(w), \psi(w)) d\rho(w)$, and then $\phi_n \rightarrow \phi$ in this topology if and only if ϕ_n converges to ϕ in measure. Then Lemma 3.2 says that for each measure preserving action $\Gamma \curvearrowright^o (W, \rho)$, the map $\phi \mapsto (\Phi^{\phi, o})_* \rho$ from $L(W, \rho, K)$ to $M_s(K^\Gamma)$ is continuous. The metric \tilde{d} is complete, and \tilde{d} is separable when (W, ρ) is standard. We note for later use that the set of all $\phi \in L(W, \rho, K)$ with finite range is dense in $L(W, \rho, K)$ (this follows from d being separable). Proofs of these facts may be found in [Ke10, §19] and [Mo76] (these references assume that the space (W, ρ) is standard, but this assumption is not used to prove the facts mentioned here).

We will find the following generalization of weak containment useful.

Definition 3.4. Let \mathcal{A} and \mathcal{B} be two sets of measure preserving actions of Γ . We say that \mathcal{A} is *weakly contained in* \mathcal{B} , written $\mathcal{A} \prec \mathcal{B}$, if for every $\Gamma \curvearrowright^a (X, \mu) = \mathbf{a} \in \mathcal{A}$, for any Borel partition A_0, \dots, A_{k-1} of X , $F \subseteq \Gamma$ finite, and $\epsilon > 0$, there exist $\Gamma \curvearrowright^b (Y, \nu) = \mathbf{b} \in \mathcal{B}$ and a Borel partition B_0, \dots, B_{k-1} of Y such that

$$|\mu(\gamma^a A_i \cap A_j) - \nu(\gamma^b B_i \cap B_j)| < \epsilon$$

for all $i, j < k$ and $\gamma \in F$.

This is a generalization of weak containment in the sense that when $\mathcal{A} = \{\mathbf{a}\}$ and $\mathcal{B} = \{\mathbf{b}\}$ are both singletons, $\mathcal{A} \prec \mathcal{B}$ if and only if $\mathbf{a} \prec \mathbf{b}$ in the original sense defined in the introduction. We write $\mathbf{a} \prec \mathcal{B}$ for $\{\mathbf{a}\} \prec \mathcal{B}$, and $\mathcal{A} \prec \mathbf{b}$ for $\mathcal{A} \prec \{\mathbf{b}\}$. If both $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{A}$ then we put $\mathcal{A} \sim \mathcal{B}$. It is clear that \prec is a reflexive and transitive relation on sets of actions. The arguments in [Ke10, Proposition 10.1] show the following proposition.

PROPOSITION 3.5. Let \mathcal{A} and \mathcal{B} be sets of non-atomic measure preserving actions of Γ . Then $\mathcal{A} \prec \mathcal{B}$ if and only if for every $\Gamma \curvearrowright^a (X, \mu) = \mathbf{a} \in \mathcal{A}$, there exists a sequence $\mathbf{a}_n \in A(\Gamma, X, \mu)$, $n \in \mathbb{N}$, converging to \mathbf{a} such that each \mathbf{a}_n is isomorphic to some $\mathbf{b}_n \in \mathcal{B}$. In particular, $\mathbf{a} \prec \mathcal{B}$ if and only if $\mathbf{a} \in \{\mathbf{d} \in A(\Gamma, X, \mu) : \exists \mathbf{b} \in \mathcal{B} \mathbf{d} \cong \mathbf{b}\}$.

We also have the corresponding generalization of [AW13, Lemma 8].

PROPOSITION 3.6. Let \mathcal{A} and \mathcal{B} be sets of measure preserving actions of Γ . Then the following are equivalent.

- (1) \mathcal{A} is weakly contained in \mathcal{B} .
- (2) $\bigcup_{\mathbf{d} \in \mathcal{A}} E(\mathbf{d}, K) \subseteq \overline{\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, K)}$ for every finite K .
- (3) $\bigcup_{\mathbf{d} \in \mathcal{A}} E(\mathbf{d}, K) \subseteq \overline{\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, K)}$ for every compact Polish space K .
- (4) $\bigcup_{\mathbf{d} \in \mathcal{A}} E(\mathbf{d}, 2^{\mathbb{N}}) \subseteq \overline{\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, 2^{\mathbb{N}})}$.

Proof. It suffices to show this for the case where $\mathcal{A} = \{\mathbf{d}\}$ is a singleton. We let (X, μ) be the space of \mathbf{d} . For each $\gamma \in \Gamma$ let $\pi_\gamma : K^\Gamma \rightarrow K$ denote the projection map $\pi_\gamma(f) = f(\gamma)$. For $\gamma \in \Gamma$, $J \subseteq \Gamma$ and $\sigma \in k^J$ let $\gamma \cdot \sigma \in k^{\gamma J}$ be given by $(\gamma \cdot \sigma)(\delta) = \sigma(\gamma^{-1}\delta)$ for each $\delta \in \gamma J$.

We begin with the implication (1) \Rightarrow (2). It suffices to show (2) for the case $K = k = \{0, 1, \dots, k-1\}$ for some $k \in \mathbb{N}$. Fix a Borel function $\phi : X \rightarrow k$, let $\lambda = (\Phi^{\phi, d})_* \mu$, and let $A_i = \phi^{-1}(\{i\})$ for $i < k$. We will show that $\lambda \in \overline{\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, K)}$. Fix an exhaustive sequence $e \in F_0 \subseteq F_1 \subseteq \dots$ of finite subsets of Γ . For each finite $F \subseteq \Gamma$ and function $\tau : F \rightarrow k$ let $A_\tau = \bigcap_{\gamma \in F} \gamma^d A_{\tau(\gamma)}$. As $\mathbf{d} \prec \mathcal{B}$ we may find for each $n \in \mathbb{N}$ a measure preserving action $\mathbf{b}_n = \Gamma \curvearrowright^{b_n} (Y_n, \nu_n)$ in \mathcal{B} along with Borel partitions $\{B_\tau^n\}_{\tau \in k^{F_n}}$ of Y_n such that

$$|\mu(\gamma^d A_{\tau_1} \cap A_{\tau_2}) - \nu_n(\gamma^{b_n} B_{\tau_1}^n \cap B_{\tau_2}^n)| < \epsilon_n \quad (3.1)$$

for all $\gamma \in F_n$, $\tau_1, \tau_2 \in k^{F_n}$, and where ϵ_n is small depending on n, k , and $|F_n|$. Define $\psi_n : Y_n \rightarrow k$ by $\psi_n(y) = i$ if $y \in B_\tau^n$ for some $\tau \in k^{F_n}$ with $\tau(e) = i$, and let $\lambda_n = (\Phi^{\psi_n, b_n})_* \nu_n$. To show that $\lambda_n \rightarrow \lambda$ it suffices to show that $\lambda_n(A) \rightarrow \lambda(A)$ for every basic clopen set $A \subseteq k^\Gamma$ of the form $A = \bigcap_{\gamma \in F} \pi_\gamma^{-1}(\{i_\gamma\})$, where $e \in F \subseteq \Gamma$ is finite and $i_\gamma < k$ for each $\gamma \in F$. We let $v \in k^F$ be the function $v(\gamma) = i_\gamma$.

For $i < k$ let $B_i^n = \bigsqcup \{B_\tau^n : \tau \in k^{F_n} \text{ and } \tau(e) = i\}$. Let n_0 be so large that $F^2 \subseteq F_{n_0}$ and for all $n > n_0$ and each $\sigma \in k^J$, $J \subseteq F_n$, let $B_\sigma^n = \bigsqcup \{B_\tau^n : \tau \in k^{F_n} \text{ and } \sigma \sqsubseteq \tau\}$ and let $\tilde{B}_\sigma^n = \bigcap_{\gamma \in J} \gamma^{b_n} B_{\sigma(\gamma)}^n$. Then $B_i^n = \bigsqcup \{B_\sigma^n : \sigma \in k^F \text{ and } \sigma(e) = i\}$. For $\sigma \in k^F$ and $\gamma \in F$ we have

$$\begin{aligned} & |\nu_n(\gamma^{b_n} B_\sigma^n \cap B_{\gamma \cdot \sigma}^n) - \mu(\gamma^d A_\sigma \cap A_{\gamma \cdot \sigma})| \\ & \leq \sum_{\{\tau \in k^{F_n} : \sigma \sqsubseteq \tau\}} \sum_{\{\tau' \in k^{F_n} : \gamma \cdot \sigma \sqsubseteq \tau'\}} |\nu_n(\gamma^{b_n} B_\tau^n \cap B_{\tau'}^n) - \mu(\gamma^d A_\tau \cap A_{\tau'})| \leq \epsilon_n k^{2|F_n|}. \end{aligned}$$

Similarly,

$$|\nu_n(B_\sigma^n) - \mu(A_\sigma)| < \epsilon_n k^{2|F_n|} \quad \text{and} \quad |\nu_n(B_{\gamma \cdot \sigma}^n) - \mu(A_{\gamma \cdot \sigma})| < \epsilon_n k^{2|F_n|}.$$

Since $\gamma^d A_\sigma = A_{\gamma \cdot \sigma}$ we obtain from this the estimate

$$d_{\nu_n}(\gamma^{b_n}(B_\sigma^n), B_{\gamma \cdot \sigma}^n) = \nu_n(B_\sigma^n) + \nu_n(B_{\gamma \cdot \sigma}^n) - 2\nu_n(\gamma^{b_n}(B_\sigma^n) \cap B_{\gamma \cdot \sigma}^n) < 3\epsilon_n k^{2|F_n|}. \quad (3.2)$$

Since $\{B_\tau^n\}_{\tau \in k^{F_n}}$ is a partition of Y_n and $F^2 \subseteq F_n$ we have the set identities

$$B_v^n = \bigsqcup_{\substack{\tau \in k^{F_n} \\ v \sqsubseteq \tau}} B_\tau^n = \bigcap_{\gamma \in F} \bigsqcup_{\substack{\sigma \in k^{\gamma F} \\ \sigma(\gamma) = v(\gamma)}} B_\sigma^n = \bigcap_{\gamma \in F} \bigsqcup_{\substack{\sigma \in k^F \\ \sigma(e) = v(\gamma)}} B_{\gamma \cdot \sigma}^n.$$

By (3.2) the d_{ν_n} -distance of this is no more than $3|F|\epsilon_n k^{3|F_n|}$ from the set

$$\bigcap_{\gamma \in F} \bigsqcup_{\substack{\sigma \in k^F \\ \sigma(e) = v(\gamma)}} \gamma^{b_n} B_\sigma^n = \bigcap_{\gamma \in F} \gamma^{b_n} B_{v(\gamma)}^n = \tilde{B}_v^n.$$

Thus

$$\begin{aligned} |\lambda_n(A) - \lambda(A)| &= |v_n(\tilde{B}_v^n) - \mu(A_v)| \leq 3|F|\epsilon_n k^{3|F_n|} + |v_n(B_v^n) - \mu(A_v)| \\ &< 3|F|\epsilon_n k^{3|F_n|} + \epsilon_n k^{2|F_n|} \rightarrow 0 \end{aligned}$$

by our choice of ϵ_n .

For the implication (2) \Rightarrow (3), let K be a compact Polish space. It follows from Lemma 3.2 and Remark 3.3 that the set $E_f(\mathbf{d}, K)$ of all measures $\lambda \in E(\mathbf{d}, K)$ coming from Borel $\phi: X \rightarrow K$ with finite range is dense in $E(\mathbf{d}, K)$. By (2) we then have

$$E_f(\mathbf{d}, K) \subseteq \overline{\bigcup_{\mathbf{b} \in \mathcal{B}} E_f(\mathbf{b}, K)} \subseteq \overline{\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, K)},$$

and (3) now follows.

The implication (3) \Rightarrow (4) is trivial.

For the implication (4) \Rightarrow (1), given a Borel partition A_0, \dots, A_{m-1} of X , $F \subseteq \Gamma$ finite, and $\epsilon > 0$, let $k_0, \dots, k_{m-1} \in 2^{\mathbb{N}}$ be distinct and define the function $\phi: X \rightarrow 2^{\mathbb{N}}$ by $\phi(x) = i$ if $x \in A_i$. Then $\lambda = (\Phi^{\phi, d})_* \mu \in E(\mathbf{d}, 2^{\mathbb{N}})$ so by (4) there exists a sequence $\Gamma \curvearrowright^{b_n} (Y_n, v_n) = \mathbf{b}_n \in \mathcal{B}$, along with $\phi_n: Y_n \rightarrow 2^{\mathbb{N}}$, such that $\lambda_n \rightarrow \lambda$, where $\lambda_n = (\Phi^{\phi_n, b_n})_* v_n$. Let C_0, \dots, C_{m-1} be disjoint clopen subsets of $2^{\mathbb{N}}$ with $k_i \in C_i$, and for each $n \in \mathbb{N}$ let $B_i^n = \phi_n^{-1}(C_i)$. Then for all $\gamma \in F$ we have

$$\begin{aligned} |\mu(\gamma^d A_i \cap A_j) - v_n(\gamma^{b_n} B_i^n \cap B_j^n)| \\ = |\lambda(\pi_\gamma^{-1}(C_i) \cap \pi_e^{-1}(C_j)) - \lambda_n(\pi_\gamma^{-1}(C_i) \cap \pi_e^{-1}(C_j))| \rightarrow 0, \end{aligned}$$

so for large enough n this quantity is smaller than ϵ . \square

3.2. Convexity in the space of actions. The convex sum of measure preserving actions is defined as follows (see also [Ke10, §10 (F)]). Let $N \in \{1, 2, \dots, \infty = \mathbb{N}\}$ and let $\alpha = (\alpha_0, \alpha_1, \dots) \in [0, 1]^N$ be a finite or countably infinite sequence of non-negative real numbers with $\sum_{i < N} \alpha_i = 1$. Given actions $\mathbf{b}_i = \Gamma \curvearrowright^{b_i} (X_i, \mu_i)$, $i < N$, we let $\sum_{i < N} X_i = \{(i, x) : i < N \text{ and } x \in X_i\}$ and we let $\tilde{\mu}_i$ be the image measure of μ_i under the inclusion map $X_i \hookrightarrow \sum_{i < N} X_i$, $x \mapsto (i, x)$. We obtain a measure preserving action $\sum_{i < N} \alpha_i \mathbf{b}_i = \Gamma \curvearrowright^{\sum_{i < N} b_i} (\sum_{i < N} X_i, \sum_{i < N} \alpha_i \tilde{\mu}_i)$ defined by $\gamma^{\sum_{i < N} b_i} \cdot (i, x) = (i, \gamma^{b_i} \cdot x)$. If $(X_i, \mu_i) = (X, \mu)$ for each $i < N$ then $(\sum_{i < N} X_i, \sum_{i < N} \alpha_i \tilde{\mu}_i) = (N \times X, \eta_\alpha \times \mu)$, where η_α is the discrete probability measure on N given by $\eta_\alpha(\{i\}) = \alpha_i$. If, furthermore, $\mathbf{b}_i = \mathbf{b}$ for each $i < N$ then $\sum_{i < N} \alpha_i \mathbf{b}_i = \iota_{\eta_\alpha} \times \mathbf{b}$ is simply the product action.

LEMMA 3.7. *Let $\mathbf{b} \in A(\Gamma, X, \mu)$ and let $\mathbf{d} = \iota_{\eta_\alpha} \times \mathbf{b} = \sum_{i=0}^{n-1} \alpha_i \mathbf{b}$. Then $E(\mathbf{d}, K) \subseteq \text{co}E(\mathbf{b}, K) \subseteq E(\iota \times \mathbf{b}, K)$ for every compact Polish K .*

Proof. Given $\phi: n \times X \rightarrow K$, we want to show that $(\Phi^{\phi, d})_*(\eta_\alpha \times \mu) \in \text{co}E(\mathbf{b}, K)$. Let $\phi_i: X \rightarrow K$ be given by $\phi_i(x) = \phi(i, x)$. Then $(\Phi^{\phi, d})^{-1}(A) = \bigsqcup_{i=0}^{n-1} \{i\} \times (\Phi^{\phi_i, b})^{-1}(A)$ for $A \subseteq K^\Gamma$ and it follows that $(\Phi^{\phi, d})_*(\eta_\alpha \times \mu) = \sum_{i=0}^{n-1} \alpha_i (\Phi^{\phi_i, b})_* \mu$, which shows the first inclusion.

Let the underlying space of ι be (Z, η) . Given Borel functions $\phi_0, \dots, \phi_{n-1} : X \rightarrow K$ and $\alpha_0, \dots, \alpha_{n-1} \geq 0$ with $\sum_{i=0}^{n-1} \alpha_i = 1$, we want to show that $\sum_{i=0}^{n-1} \alpha_i (\Phi^{\phi_i, b})_* \mu \in E(\iota \times \mathbf{b}, K)$. Let C_0, \dots, C_{n-1} be a Borel partition of Z with $\eta(C_i) = \alpha_i$ for $i = 0, \dots, n-1$. Define $i : Z \rightarrow n$ by $i(z) = i$ if $z \in C_i$ and let $\phi : Z \times X \rightarrow K$ be the map $\phi(z, x) = \phi_{i(z)}(x)$. Then

$$\Phi^{\phi, \iota \times b}(z, x)(\gamma) = \phi(\gamma^{\iota \times b} \cdot (z, x)) = \phi(z, \gamma^b \cdot x) = \phi_{i(z)}(\gamma^b \cdot x) = \Phi^{\phi_{i(z)}, b}(x)(\gamma),$$

and so $(\Phi^{\phi, \iota \times b})^{-1}(A) = \bigsqcup_{i=0}^{n-1} C_i \times (\Phi^{\phi_i, b})^{-1}(A)$ for all $A \subseteq K^\Gamma$. It now follows that $\sum_{i=0}^{n-1} \alpha_i (\Phi^{\phi_i, b})_* \mu = (\Phi^{\phi, \iota \times b})_*(\eta \times \mu)$. \square

LEMMA 3.8. Let $\mathbf{b} \in A(\Gamma, X, \mu)$, let $\alpha(n) = (1/n, \dots, 1/n) \in [0, 1]^n$, and let

$$\mathcal{B}_1 = \{\iota_{\alpha(n)} \times \mathbf{b} : n \geq 1\}, \quad \mathcal{B}_2 = \left\{ \iota_{\eta_\alpha} \times \mathbf{b} : n \geq 1, \alpha \in [0, 1]^n, \sum_{i=0}^{n-1} \alpha_i = 1 \right\}.$$

Then $\iota \times \mathbf{b} \sim \mathcal{B}_1 \sim \mathcal{B}_2$.

Proof. $\mathcal{B}_1 \prec \mathcal{B}_2$ is trivial. $\mathcal{B}_2 \prec \iota \times \mathbf{b}$ is clear (in fact, $\mathbf{d} \sqsubseteq \iota \times \mathbf{b}$ for every $\mathbf{d} \in \mathcal{B}_2$). It remains to show that $\iota \times \mathbf{b} \prec \mathcal{B}_1$. Let (Z, η) be the underlying non-atomic probability space of ι and let $\lambda = \eta \times \mu$. Fix a partition \mathcal{P} of $Z \times X$, $F \subseteq \Gamma$ finite and $\epsilon > 0$. We may assume without loss of generality that \mathcal{P} is of the form $\mathcal{P} = \{A_i \times B_j : 0 \leq i < n, 0 \leq j < m\}$, where $\{A_i\}_{i=0}^{n-1}$ is a partition of Z , $\{B_j\}_{j=0}^{m-1}$ is a partition of X , and all the sets A_0, \dots, A_{n-1} have equal measure. Let $C_{i,j} = \{(i, x) \in n \times X : x \in B_j\}$. Then, letting $\mathbf{d} = \iota_{\alpha(n)} \times \mathbf{b}$, for all $\gamma \in F$ and $i, i' \leq n, j, j' \leq m$, if $i \neq i'$ we have $\gamma^{\mathbf{d}} C_{i,j} \cap C_{i',j'} = \emptyset = \gamma^{\iota \times b}(A_i \times B_j) \cap (A_{i'} \times B_{j'})$, while if $i = i'$ we have

$$\begin{aligned} (\eta_{\alpha(n)} \times \mu)(\gamma^{\mathbf{d}} C_{i,j} \cap C_{i,j'}) &= \frac{1}{n} \mu(\gamma^b B_j \cap B_{j'}) \\ &= \eta(A_i) \mu(\gamma^b B_j \cap B_{j'}) = \lambda(\gamma^{\iota \times b}(A_i \times B_j) \cap (A_i \times B_{j'})), \end{aligned}$$

showing that $\iota \times \mathbf{b} \prec \mathcal{B}_1$. \square

Proof of Theorem 1.1. We apply Proposition 3.6 and Lemma 3.8, then Lemma 3.7 to obtain

$$E(\iota \times \mathbf{b}, K) \subseteq \overline{\bigcup_{n \geq 1} E(\iota_{\eta_{\alpha(n)}} \times \mathbf{b}, K)} \subseteq \overline{\text{co} E(\mathbf{b}, K)} \subseteq \overline{E(\iota \times \mathbf{b}, K)}$$

and so $\overline{E(\iota \times \mathbf{b}, K)} = \overline{\text{co} E(\mathbf{b}, K)}$. \square

The proof of Theorem 1.2 now proceeds in analogy with the proof of the corresponding fact for unitary representations (see [BdlHV08, F.1.4]).

Proof of Theorem 1.2. Suppose that \mathbf{a} is ergodic and $\mathbf{a} \prec \iota \times \mathbf{b}$. We want to show that $\mathbf{a} \prec \mathbf{b}$, or equivalently $E(\mathbf{a}, K) \subseteq \overline{E(\mathbf{b}, K)}$ for every compact Polish space K . By hypothesis we have that $E(\mathbf{a}, K) \subseteq \overline{E(\iota \times \mathbf{b}, K)}$, so by Theorem 1.1, $E(\mathbf{a}, K) \subseteq \overline{\text{co} E(\mathbf{b}, K)}$. Since every element of $E(\mathbf{a}, K)$ is ergodic, $E(\mathbf{a}, K)$ is contained in the extreme points of $M_s(K^\Gamma)$, and so *a fortiori* $E(\mathbf{a}, K)$ is contained in the extreme points of $\overline{\text{co} E(\mathbf{b}, K)}$. Since in a locally convex space the extreme points of a given compact convex set are contained in every closed set generating that convex set (see, for example, [Ph02, Proposition 1.5]), it follows that $E(\mathbf{a}, K) \subseteq \overline{E(\mathbf{b}, K)}$, as was to be shown. \square

3.3. *Ergodic decomposition and weak containment.* We begin with the following observation about factors.

PROPOSITION 3.9. *Let \mathbf{d} be a measure preserving action of Γ on (Y, ν) and suppose that $\pi : (Y, \nu) \rightarrow (Z, \eta)$ is a factor map from \mathbf{d} onto an identity action $\Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$. Let $\nu = \int_Z \nu_z d\eta$ be the disintegration of ν with respect to π and let $\mathbf{d}_z = \Gamma \curvearrowright^{\mathbf{d}} (Y, \nu_z)$. Suppose that $\mathbf{a} = \Gamma \curvearrowright^{\mathbf{a}} (X, \mu)$ is an ergodic factor of \mathbf{d} via the map $\varphi : (Y, \nu) \rightarrow (X, \mu)$. Then for η -almost every $z \in Z$, \mathbf{a} is a factor of \mathbf{d}_z via the map φ .*

Proof. The map $\pi \times \varphi : (Y, \nu) \rightarrow (Z \times X, (\pi \times \varphi)_* \nu)$, $y \mapsto (\pi(y), \varphi(y))$, factors \mathbf{d} onto a joining \mathbf{b} of the identity action ι_η and the ergodic action \mathbf{a} . Since ergodic and identity actions are disjoint [GI03, 6.24] we have that $(\pi \times \varphi)_* \nu = \eta \times \mu$ and $\mathbf{b} = \iota_\eta \times \mathbf{a}$. The measure $(\pi \times \varphi)_* \nu_z$ lives on $\{z\} \times X$ almost surely, and

$$\eta \times \mu = (\pi \times \varphi)_* \nu = \int_Z (\pi \times \varphi)_* \nu_z d\eta,$$

so by uniqueness of disintegration $(\pi \times \varphi)_* \nu_z = \delta_z \times \mu$ almost surely. Since $\text{proj}_X \circ (\pi \times \varphi) = \varphi$ we have that $\varphi_* \nu_z = (\text{proj}_X)_* (\delta_z \times \mu) = \mu$ almost surely. \square

COROLLARY 3.10. *If \mathbf{a} is ergodic and φ factors \mathbf{d} onto \mathbf{a} then φ factors almost every ergodic component of \mathbf{d} onto \mathbf{a} .*

Using ultraproducts of measure preserving actions (see Appendix A) we can prove an analogous result for weak containment, which generalizes Theorem 1.2. For the remainder of this section we fix a non-principal ultrafilter \mathcal{U} on \mathbb{N} and we also fix a compact Polish space K homeomorphic to $2^{\mathbb{N}}$. Let $\mathbf{a}_n = \Gamma \curvearrowright^{a_n} (Y_n, \nu)$, $n \in \mathbb{N}$, be a sequence of measure preserving actions of Γ and let $\mathbf{a}_{\mathcal{U}} = \Gamma \curvearrowright^{a_{\mathcal{U}}} (Y_{\mathcal{U}}, \mu_{\mathcal{U}})$ be the ultraproduct of the sequence \mathbf{a}_n with respect to the non-principal ultrafilter \mathcal{U} on \mathbb{N} . Let $\phi_n : Y_n \rightarrow K$ be a sequence of Borel functions and let $\Phi_n = \Phi^{\phi_n, a_n} : Y_n \rightarrow K^{\Gamma}$. We let ϕ denote the ultralimit function determined by the sequence ϕ_n , that is, $\phi : Y_{\mathcal{U}} \rightarrow K$ is the function given by

$$\phi([y_n]) = \lim_{n \rightarrow \mathcal{U}} \phi_n(y_n)$$

for $[y_n] \in Y_{\mathcal{U}}$. The function ϕ is $\mathbf{B}_{\mathcal{U}}$ -measurable since $\phi^{-1}(V) = [\phi_n^{-1}(V)]$ whenever $V \subseteq K$ is open.

PROPOSITION 3.11. *Let $\Phi = \Phi^{\phi, a_{\mathcal{U}}}$. Then:*

- (1) $\Phi([y_n]) = \lim_{n \rightarrow \mathcal{U}} \Phi_n(y_n)$ for all $[y_n] \in Y_{\mathcal{U}}$;
- (2) $\Phi_* \nu_{\mathcal{U}} = \lim_{n \rightarrow \mathcal{U}} (\Phi_n)_* \nu_n$;
- (3) for every $\mathbf{B}_{\mathcal{U}}$ -measurable function $\psi : Y_{\mathcal{U}} \rightarrow K$ there exists a sequence $\varphi_n : Y_n \rightarrow K$ of Borel functions such that $\psi([y_n]) = \lim_{n \rightarrow \mathcal{U}} \varphi_n(y_n)$ for $\nu_{\mathcal{U}}$ -almost every $[y_n] \in Y_{\mathcal{U}}$.

Proof. (1) For each $[y_n] \in Y_{\mathcal{U}}$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} \Phi([y_n])(\gamma) &= \phi((\gamma^{-1})^{a_{\mathcal{U}}} [y_n]) = \phi([(\gamma^{-1})^{a_n} y_n]) = \lim_{n \rightarrow \mathcal{U}} \phi((\gamma^{-1})^{a_n} y_n) \\ &= \lim_{n \rightarrow \mathcal{U}} \Phi_n(y_n)(\gamma) = \left(\lim_{n \rightarrow \mathcal{U}} \Phi_n(y_n) \right)(\gamma), \end{aligned}$$

the last equality following by continuity of the evaluation map $f \mapsto f(\gamma)$ on K^{Γ} .

(2) Let $\lambda = \lim_{n \rightarrow \mathcal{U}} (\Phi_n)_* \nu_n$. Then λ is the unique element of $M_s(K^\Gamma)$ such that $\lambda(C) = \lim_{n \rightarrow \mathcal{U}} ((\Phi_n)_* \nu_n(C))$ for all clopen $C \subseteq K^\Gamma$. Part (1) implies that $\Phi^{-1}(C) = [\Phi_n^{-1}(C)]$ whenever $C \subseteq K^\Gamma$ is clopen, and so $\Phi_* \nu_{\mathcal{U}}(C) = \lim_{n \rightarrow \mathcal{U}} \nu_n(\Phi_n^{-1}(C)) = \lim_{n \rightarrow \mathcal{U}} ((\Phi_n)_* \nu_n(C))$.

(3) We may assume that $K = 2^{\mathbb{N}}$. For $m \in \mathbb{N}$ define $\psi_m : Y_{\mathcal{U}} \rightarrow K$ by $\psi_m([y_n]) = \psi([y_n])(m)$. For $i \in \{0, 1\}$ let $A_n^{m,i} = \psi_m^{-1}(\{i\}) \in \mathcal{B}_{\mathcal{U}}$ and fix $[A_n^{m,i}] \in \mathcal{A}_{\mathcal{U}}$ such that $\nu_{\mathcal{U}}(A_n^{m,i} \Delta [A_n^{m,i}]) = 0$. For each $m, n \in \mathbb{N}$ let $B_n^{m,0} = A_n^{m,0} \setminus A_n^{m,1}$ and let $B_n^{m,1} = Y_n \setminus B_n^{m,0}$ so that $\{B_n^{m,0}, B_n^{m,1}\}$ is a Borel partition of Y_n . Then for each $m \in \mathbb{N}$ we have $\nu_{\mathcal{U}}(A_n^{m,0} \Delta [B_n^{m,0}]) = 0 = \nu_{\mathcal{U}}(A_n^{m,1} \Delta [B_n^{m,1}])$. Define $\varphi_n : Y_n \rightarrow K$ by taking $\varphi_n(y)(m) = i$ if and only if $y \in B_n^{m,i}$. Let $\varphi : Y_{\mathcal{U}} \rightarrow K$ be the ultralimit function $\varphi([y_n]) = \lim_{n \rightarrow \mathcal{U}} \varphi_n(y_n)$. Then for $i \in \{0, 1\}$ we have

$$\varphi([y_n])(m) = i \Leftrightarrow \lim_{n \rightarrow \mathcal{U}} (\varphi_n(y_n)(m)) = i \Leftrightarrow \{n : y_n \in B_n^{m,i}\} \in \mathcal{U} \Leftrightarrow [y_n] \in [B_n^{m,i}],$$

and so φ is equal to ψ off the null set $\bigcup_{m \in \mathbb{N}, i \in \{0,1\}} A_n^{m,i} \Delta [B_n^{m,i}]$. \square

THEOREM 3.12. *Let \mathbf{d} be a measure preserving action of Γ on (Y, ν) and suppose that $\pi : (Y, \nu) \rightarrow (Z, \eta)$ is a factor map from \mathbf{d} onto an identity action $\Gamma \curvearrowright^{\eta} (Z, \eta)$. Let $\nu = \int_z \nu_z d\eta$ be the disintegration of ν with respect to π and let $\mathbf{d}_z = \Gamma \curvearrowright^{\mathbf{d}} (Y, \nu_z)$. Suppose that $\mathbf{a} = \Gamma \curvearrowright^{\mathbf{a}} (X, \mu)$ is ergodic and weakly contained in \mathbf{d} . Then \mathbf{a} is weakly contained in \mathbf{d}_z for almost all $z \in Z$.*

Proof. Taking $K = 2^{\mathbb{N}}$, it suffices to show for each $\lambda \in E(\mathbf{a}, K)$ that $\eta(\{z : \lambda \in \overline{E(\mathbf{d}_z, K)}\}) = 1$. Let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} and let $\mathbf{d}_{\mathcal{U}} = \Gamma \curvearrowright^{\mathbf{d}_{\mathcal{U}}} (Y_{\mathcal{U}}, \nu_{\mathcal{U}})$ and $\mathbf{u}_{\mathcal{U}} = \Gamma \curvearrowright^{\mathbf{u}_{\mathcal{U}}} (Z_{\mathcal{U}}, \eta_{\mathcal{U}})$ be the ultrapowers of \mathbf{d} and \mathbf{u}_{η} respectively. The map $\pi_{\mathcal{U}} : Y_{\mathcal{U}} \rightarrow Z_{\mathcal{U}}$ defined by $\pi_{\mathcal{U}}([y_n]) = [\pi(y_n)]$ factors $\mathbf{d}_{\mathcal{U}}$ onto $\mathbf{u}_{\mathcal{U}}$.

Given any $\lambda \in E(\mathbf{a}, K)$, since $\mathbf{a} \prec \mathbf{d}$ there exists $\phi_n : Y \rightarrow K$ such that $(\Phi_n^{\phi_n, \mathbf{d}})_* \nu \rightarrow \lambda$. Let $\phi : Y_{\mathcal{U}} \rightarrow K$ be the ultralimit of the functions ϕ_n , let $\Phi_n = \Phi_n^{\phi_n, \mathbf{d}}$, and let $\Phi = \Phi^{\phi, \mathbf{d}_{\mathcal{U}}} : Y_{\mathcal{U}} \rightarrow K^\Gamma$. By Proposition 3.11(2), Φ factors $\mathbf{d}_{\mathcal{U}}$ onto $\Gamma \curvearrowright^s (K^\Gamma, \lambda)$.

Define the map $\pi_{\mathcal{U}} \times \Phi : Y_{\mathcal{U}} \rightarrow Z_{\mathcal{U}} \times K^\Gamma$ by $(\pi_{\mathcal{U}} \times \Phi)([y_n]) := (\pi_{\mathcal{U}}([y_n]), \Phi([y_n]))$. Then the measure $(\pi_{\mathcal{U}} \times \Phi)_* \nu_{\mathcal{U}}$ is a joining of $\mathbf{u}_{\mathcal{U}}$ with $\Gamma \curvearrowright^s (K^\Gamma, \lambda)$, hence $(\pi_{\mathcal{U}} \times \Phi)_* \nu_{\mathcal{U}} = \eta_{\mathcal{U}} \times \lambda$ since identity systems and ergodic systems are disjoint (note that disjointness of identity and ergodic systems which are not necessarily standard follows from disjointness of such systems in the standard case). Now let $\nu_{[z_n]} = \prod_n \nu_{z_n} / \mathcal{U}$, so that $\nu_{[z_n]}$ is a probability measure on $\mathcal{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ for all $[z_n] \in Z_{\mathcal{U}}$.

CLAIM. $\lim_{n \rightarrow \mathcal{U}} (\Phi_n)_* \nu_{z_n} = \lambda$ for $\eta_{\mathcal{U}}$ -almost every $[z_n] \in Z_{\mathcal{U}}$.

Proof of claim. By Proposition A.1, $\nu_{\mathcal{U}}(A) = \int_{[z_n]} \nu_{[z_n]}(A) d\eta_{\mathcal{U}}$ for all $A \in \mathcal{B}_{\mathcal{U}}(Y_{\mathcal{U}})$. As $\sigma_* \nu_{[z_n]}$ lives on $\{[z_n]\} \times K^\Gamma$ it follows for $D \in \mathcal{B}_{\mathcal{U}}(Z_{\mathcal{U}})$ and $C \subseteq K^\Gamma$ clopen that

$$\begin{aligned} \int_{[z_n] \in D} \lambda(C) d\eta_{\mathcal{U}} &= \eta_{\mathcal{U}}(D) \lambda(C) = \rho(D \times C) = \int_{[z_n]} \sigma_* \nu_{[z_n]}(D \times C) d\eta_{\mathcal{U}} \\ &= \int_{[z_n] \in D} \sigma_* \nu_{[z_n]}(Z_{\mathcal{U}} \times C) d\eta_{\mathcal{U}} = \int_{[z_n] \in D} \Phi_* \nu_{[z_n]}(C) d\eta_{\mathcal{U}}. \end{aligned}$$

Thus for each clopen $C \subseteq K^\Gamma$, $\Phi_* \nu_{[z_n]}(C) = \lambda(C)$ for $\eta_{\mathcal{U}}$ almost every $[z_n] \in Z_{\mathcal{U}}$. It follows that $\Phi_* \nu_{[z_n]} = \lambda$ for $\eta_{\mathcal{U}}$ -almost every $[z_n] \in Z_{\mathcal{U}}$. By Proposition 3.11(2), $\lim_{n \rightarrow \mathcal{U}} (\Phi_n)_* \nu_{z_n} = \lambda$ for $\eta_{\mathcal{U}}$ -almost every $[z_n] \in Z_{\mathcal{U}}$. \square

If now V is any open neighborhood of λ in $M_s(K^\Gamma)$ then let $B = \{z \in Z : E(\mathbf{d}_z, K) \cap V \neq \emptyset\}$. Suppose toward a contradiction that $\eta(B) > 0$, then let $B_n = B$ for all n so that $[B_n] \in \mathcal{A}_U(Z_U)$ and $\eta_U([B_n]) > 0$. Thus, for some $[z_n] \in [B_n]$ we have $\lim_{n \rightarrow \infty} (\Phi_n)_* \nu_{z_n} = \lambda$ and so for some $n \in \mathbb{N}$ with $z_n \in B_n = B$ we have $(\Phi_n)_* \nu_{z_n} \in E(\mathbf{d}_{z_n}, K) \cap V$, which contradicts the definition of B . This shows that $\eta(B) = 0$. It follows that $\lambda \in \overline{E(\mathbf{d}_z, K)}$ almost surely. \square

THEOREM 3.13. *Let $\varphi : \Gamma \curvearrowright^b (X, \mu) \rightarrow \Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$ and $\psi : \Gamma \curvearrowright^d (Y, \nu) \rightarrow \Gamma \curvearrowright^{\iota_\eta} (Z, \eta)$ be factor maps from \mathbf{b} and \mathbf{d} respectively onto ι_η . Let $\mu = \int_Z \mu_z d\eta$ and $\nu = \int_Z \nu_z d\eta$ be the disintegrations of μ and ν via φ and ψ respectively, and for each $z \in Z$ let $\mathbf{b}_z = \Gamma \curvearrowright^b (X, \mu_z)$ and let $\mathbf{d}_z = \Gamma \curvearrowright^d (Y, \nu_z)$. Then:*

- (1) *if $\mathbf{b}_z \prec \mathbf{d}_z$ for all $z \in Z$ then $\mathbf{b} \prec \mathbf{d}$;*
- (2) *if $\mathbf{b} \prec \mathbf{d}_z$ for all $z \in Z$ then $\iota_\eta \times \mathbf{b} \prec \mathbf{d}$, and if $\mathbf{b}_z \prec \mathbf{d}$ for all $z \in Z$ then $\mathbf{b} \prec \iota_\eta \times \mathbf{d}$;*
- (3) *if $\mathbf{b}_z \sim \mathbf{d}_z$ for all $z \in Z$ then $\mathbf{b} \sim \mathbf{d}$, and if $\mathbf{b} \sim \mathbf{d}_z$ for all $z \in Z$ then $\iota_\eta \times \mathbf{b} \sim \mathbf{d}$.*

We also have the following version for stable weak containment (see Appendix B).

- (4) *If $\mathbf{b}_z \prec_s \mathbf{d}_z$ for all $z \in Z$ then $\mathbf{b} \prec_s \mathbf{d}$.*
- (5) *If $\mathbf{b}_z \prec_s \mathbf{d}$ for all $z \in Z$ then $\mathbf{b} \prec_s \mathbf{d}$, and if $\mathbf{b}_z \prec_s \mathbf{d}$ for all $z \in Z$ then $\mathbf{b} \prec_s \mathbf{d}$.*
- (6) *If $\mathbf{b}_z \sim_s \mathbf{d}_z$ for all $z \in Z$ then $\mathbf{b} \sim_s \mathbf{d}$, and if $\mathbf{b} \sim_s \mathbf{d}_z$ for all $z \in Z$ then $\mathbf{b} \sim_s \mathbf{d}$.*

Proof. (1) Let $\{B_n\}_{n \in \mathbb{N}}$ be a countable algebra of subsets of Y generating the Borel σ -algebra of Y . Fix a partition A_0, \dots, A_{k-1} of Borel subsets of X along with $F \subseteq \Gamma$ finite and $\epsilon > 0$. For each z there exists a k -tuple $(n_0, \dots, n_{k-1}) \in \mathbb{N}^k$ such that the sets $B_{n_0}, \dots, B_{n_{k-1}} \subseteq Y$ witness that $\mathbf{b}_z \prec \mathbf{d}_z$ with respect to the parameters A_0, \dots, A_{k-1} , F , and ϵ . We let $n(z) = (n_0(z), \dots, n_{k-1}(z))$ be the lexicographically least k -tuple that satisfies this for z . For each $j < k$ the set

$$D_j = \{y \in Y : \exists z \in Z (\psi(y) = z \text{ and } y \in B_{n_j(z)})\} = \bigsqcup_z (B_{n_j(z)} \cap \psi^{-1}(z))$$

is analytic and so is measurable. For all $z \in Z$, $\gamma \in \Gamma$, and $j < k$ we then have that $\gamma^d D_j \cap \psi^{-1}(z) = \gamma^{d_z} B_{n_j(z)} \cap \psi^{-1}(z)$ and it follows that $\nu_z(\gamma^d D_j \cap D_{j'}) = \nu_z(\gamma^{d_z} B_{n_j(z)} \cap B_{n_{j'}(z)})$, since ν_z concentrates on $\psi^{-1}(z)$. For $\gamma \in F$ and $i, j < k$ we then have

$$\begin{aligned} & |\nu(\gamma^d D_i \cap D_j) - \mu(\gamma^b A_i \cap A_j)| \\ &= \left| \int_{z \in Z} \nu_z(\gamma^d D_i \cap D_j) d\eta(z) - \int_{z \in Z} \mu_z(\gamma^b A_i \cap A_j) d\eta(z) \right| \\ &\leq \int_{z \in Z} |\nu_z(\gamma^{d_z} B_{n_i(z)} \cap B_{n_j(z)}) - \mu_z(\gamma^{b_z} A_i \cap A_j)| d\eta(z) \leq \eta(Z)\epsilon = \epsilon \end{aligned}$$

which finishes the proof of (1).

Statements (2) through (6) now follow from (1). \square

Question 3.14. Is every measure preserving action \mathbf{d} of Γ stably weakly equivalent to an action with countable ergodic decomposition?

A positive answer to Question 3.14 would be an ergodic theoretic analogue of the fact that every unitary representation of Γ on a separable Hilbert space is weakly

equivalent to a countable direct sum of irreducible representations ([Di77]; this also follows from [BdlHV08, F.2.7]). We also mention the following related problem.

Problem 3.15. Describe the set $\text{ex}(\overline{\text{co}}E(\mathbf{a}, 2^{\mathbb{N}}))$ of extreme points of $\overline{\text{co}}E(\mathbf{a}, 2^{\mathbb{N}})$ for $\mathbf{a} \in A(\Gamma, X, \mu)$.

4. Consequences of Theorem 1.2 and applications to MD and EMD

4.1. *Free, non-ergodic weak equivalence classes.* We can now prove Theorem 1.3.

Proof of Theorem 1.3. If \mathbf{a} is any ergodic action of Γ and $\mathbf{a} \prec \iota \times \mathbf{b}$ then by Theorem 1.2 $\mathbf{a} \prec \mathbf{b}$, and so \mathbf{a} is strongly ergodic. It follows that we cannot also have $\iota \times \mathbf{b} \prec \mathbf{a}$, otherwise \mathbf{a} would not be strongly ergodic. \square

Remark 4.1. Foreman and Weiss [FW04, Claim 18] show that for any free measure preserving action $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ of an infinite amenable group, $\mathbf{b} \prec \mathbf{a}$ for every $\mathbf{b} \in A(\Gamma, X, \mu)$. We note that a quick alternative proof of this follows from [BT-D13, Theorem 1.2], which says that if Δ is a normal subgroup of a countably infinite group Γ and Γ/Δ is amenable, then $\mathbf{b} \prec \text{CInd}_{\Delta}^{\Gamma}((\iota \times \mathbf{b})|_{\Delta})$ for every $\mathbf{b} \in A(\Gamma, X, \mu)$. Taking Γ to be an infinite amenable group and $\Delta = \langle e \rangle$ the trivial group, the restriction $(\iota \times \mathbf{b})|_{\langle e \rangle}$ is trivial, so $\text{CInd}_{\langle e \rangle}^{\Gamma}((\iota \times \mathbf{b})|_{\langle e \rangle})$ is the Bernoulli shift action s_{Γ} of Γ . Thus, $\mathbf{b} \prec s_{\Gamma}$. By [AW13, Theorem 1] (or alternatively, Corollary 1.6), since \mathbf{a} is free, we have $s_{\Gamma} \prec \mathbf{a}$ and so $\mathbf{b} \prec \mathbf{a}$.

Combining this with Theorem 1.3 gives a new characterization of (non-)amenability for a countable group Γ .

COROLLARY 4.2. *A countably infinite group Γ is non-amenable if and only if there exists a free measure preserving action of Γ that is not weakly equivalent to any ergodic action.*

Remark 4.3. It is noted in [CK13, 4.(C)] that if Γ is a non-amenable group, and if $S \subseteq \Gamma$ is a set of generators for Γ such that the Cayley graph $\text{Cay}(\Gamma, S)$ is bipartite, then there are continuum many weak equivalence classes of free measure preserving actions of Γ . Their method of using convex combinations of actions can be used to show that this holds for *all* non-amenable Γ , and in fact the proof shows that there exists a collection $\{\mathbf{a}_{\alpha} : 0 < \alpha \leq \frac{1}{2}\}$ with \mathbf{a}_{α} and \mathbf{a}_{β} weakly incomparable when $\alpha \neq \beta$. Indeed, if $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ is any free strongly ergodic action of Γ (which exists when Γ is non-amenable), then for any $0 < \alpha < \beta \leq \frac{1}{2}$ the actions $\mathbf{a}_{\alpha} = \alpha \mathbf{a} + (1 - \alpha) \mathbf{a}$ and $\mathbf{a}_{\beta} = \beta \mathbf{a} + (1 - \beta) \mathbf{a}$ are weakly incomparable. To see this, note that any action weakly containing \mathbf{a}_{α} has a sequence of asymptotically invariant sets with measures converging to α . Since \mathbf{a} is strongly ergodic it is clear that no such sequence exists for \mathbf{a}_{β} , and so $\mathbf{a}_{\alpha} \not\prec \mathbf{a}_{\beta}$. Similarly, $\mathbf{a}_{\beta} \not\prec \mathbf{a}_{\alpha}$.

It is open whether every non-amenable group has continuum many weak equivalence classes of free *ergodic* measure preserving actions. It is in fact unknown whether there exists a non-amenable group with just one free ergodic action up to weak equivalence (though it is shown in the fourth remark after [Ke10, Proposition 13.2] that any such group must, among other things, have property (T) and cannot contain a non-abelian free group). In [AE12] Abért and Elek show that Γ has continuum many pairwise weakly incomparable (hence inequivalent) free ergodic actions when Γ is a finitely generated free group or a

linear group with property (T). Their result also holds for stable weak equivalence in view of the following consequence of Theorem 1.2.

COROLLARY 4.4. *Let \mathbf{a} and \mathbf{b} be ergodic measure preserving actions of Γ and let (Z, η) be a standard probability space. Then $\mathbf{a} \sim \mathbf{b}$ if and only if $\iota_\eta \times \mathbf{a} \sim \iota_\eta \times \mathbf{b}$. In particular, $\mathbf{a} \sim \mathbf{b}$ if and only if $\mathbf{a} \sim_s \mathbf{b}$.*

Proof. If $\mathbf{a} \sim \mathbf{b}$ then $\iota_\eta \times \mathbf{a} \sim \iota_\eta \times \mathbf{b}$ by continuity of the product operation. Conversely, if $\iota_\eta \times \mathbf{a} \sim \iota_\eta \times \mathbf{b}$ then $\mathbf{a} \prec \iota_\eta \times \mathbf{a} \prec \iota_\eta \times \mathbf{b}$ so that $\mathbf{a} \prec \mathbf{b}$ by Theorem 1.2. Likewise, $\mathbf{b} \prec \mathbf{a}$, so $\mathbf{a} \sim \mathbf{b}$. \square

I also do not know whether every non-amenable group has continuum many stable weak equivalence classes of free measure preserving actions, or whether there exists a non-amenable group all of whose free measure preserving actions are stably weakly equivalent.

4.2. The properties MD and EMD.

Definition 4.5. Let \mathcal{B} be a class of measure preserving actions of a countable group Γ . If $\mathbf{a} \in \mathcal{B}$ then \mathbf{a} is called *universal for \mathcal{B}* if $\mathbf{b} \prec \mathbf{a}$ for every $\mathbf{b} \in \mathcal{B}$. When \mathbf{a} is universal for the class of all measure preserving actions of Γ then \mathbf{a} is simply called *universal*.

Definition 4.6. [Ke12] Let Γ be a countably infinite group. Then Γ is said to have property EMD if the measure preserving action \mathbf{p}_Γ of Γ on its profinite completion is universal. Γ is said to have property EMD* if \mathbf{p}_Γ is universal for the class of all ergodic measure preserving actions of Γ . Γ is said to have property MD if $\iota \times \mathbf{p}_\Gamma$ is universal.

If Γ has property EMD, EMD*, or MD, then \mathbf{p}_Γ must be free (this follows, for example, from the proof of Lemma 5.3 below) and so Γ must be residually finite. It is also clear that EMD implies both EMD* and MD. We now show that EMD* and MD are equivalent.

Proof of Theorem 1.4. The implication $\text{EMD}^* \Rightarrow \text{MD}$ is shown in [Ke12], but also follows from Theorem 3.13 above. For the converse, suppose that Γ has MD so that $\iota \times \mathbf{p}_\Gamma$ is universal, and let \mathbf{a} be an ergodic action of Γ . Then $\mathbf{a} \prec \iota \times \mathbf{p}_\Gamma$, so since \mathbf{a} is ergodic, Theorem 1.2 implies $\mathbf{a} \prec \mathbf{p}_\Gamma$. Thus \mathbf{p}_Γ is universal for ergodic actions of Γ , and so Γ has EMD*. \square

COROLLARY 4.7. *EMD and MD are equivalent for groups without property (T).*

Proof. Suppose that Γ has MD and does not have (T). Then $\iota \times \mathbf{p}_\Gamma$ is universal and, by Theorem 1.4, \mathbf{p}_Γ is universal for ergodic measure preserving actions. Since Γ does not have property (T) there exists an ergodic $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ with $\iota \prec \mathbf{a}$, and so $\iota \prec \mathbf{a} \prec \mathbf{p}_\Gamma$. Since \mathbf{p}_Γ is ergodic with $\iota \prec \mathbf{p}_\Gamma$ it follows that $\iota \times \mathbf{p}_\Gamma \prec \mathbf{p}_\Gamma$ (see [AW13, Theorem 3]) and so \mathbf{p}_Γ is universal. \square

In what follows, if $\varphi: \Gamma \rightarrow \Delta$ is a group homomorphism then for each $\mathbf{a} \in A(\Delta, X, \mu)$ we let $\mathbf{a}^\varphi \in A(\Gamma, X, \mu)$ denote the action that is the composition of \mathbf{a} with φ , that is, $\gamma^{\mathbf{a}^\varphi} = \varphi(\gamma)^{\mathbf{a}}$. Also, we note that for any two countable groups Γ_1, Γ_2 , there is a natural equivariant homeomorphism from the diagonal action $\text{Aut}(X, \mu) \curvearrowright A(\Gamma_1, X, \mu) \times A(\Gamma_2, X, \mu)$ to $\text{Aut}(X, \mu) \curvearrowright A(\Gamma_1 * \Gamma_2, X, \mu)$. We denote this map

by $(a_1, a_2) \mapsto a_1 * a_2$. See [Ke10, §10 (G)]. We also refer to [Ke10, Appendix G] and [Zi84] for information about induced actions.

THEOREM 4.8. *Suppose that Γ_1 and Γ_2 are non-trivial countable groups and that for each $i \in \{1, 2\}$, Γ_i is either finite or has property MD. Then $\Gamma_1 * \Gamma_2$ has property EMD.*

Proof. Let $a_1 * a_2 \in A(\Gamma_1 * \Gamma_2, X, \mu)$ be given and let $U = U_1 * U_2$ be an open neighborhood of $a_1 * a_2$, where U_i is an open neighborhood of a_i for $i = 1, 2$. It suffices to show that $U_1 * U_2$ contains an ergodic profinite action. By [Ke12, Proposition 4.8], for each $i = 1, 2$ there exists a finite group $F_i \neq \{e\}$ along with a homomorphism $\varphi_i : \Gamma_i \rightarrow F_i$ and $b_i \in A(F_i, X, \mu)$ such that the corresponding measure preserving action $b_i^{\varphi_i}$ of Γ_i is in U_i . Let $\varphi = \varphi_1 * \varphi_2 : \Gamma_1 * \Gamma_2 \rightarrow F_1 * F_2$ and let $b = b_1 * b_2$. Then $b^\varphi = b_1^{\varphi_1} * b_2^{\varphi_2} \in U_1 * U_2$. For $i \in \{1, 2\}$, using continuity of the map $a \mapsto a^{\varphi_i}$, we may find an open neighborhood $V_i \subseteq A(F_i, X, \mu)$ of b_i such that $\{a^{\varphi_i} : a \in V_i\} \subseteq U_i$. Then $b \in V_1 * V_2$, and for all $d \in V_1 * V_2$ we have $d^\varphi \in U_1 * U_2$.

Fix a (possibly abelian) finite index free subgroup \mathbb{F} of $F_1 * F_2$ (one may take $\mathbb{F} = \ker(\psi) = [F_1, F_2]$, where ψ is the natural homomorphism $F_1 * F_2 \rightarrow F_1 \times F_2$), and since \mathbb{F} has EMD [Ke12, Theorem 1] we have $b|_{\mathbb{F}} \prec p_{\mathbb{F}}$. Letting $a_{F/\mathbb{F}}$ denote the action of F on F/\mathbb{F} with normalized counting measure, we now have

$$b \subseteq b \times a_{F/\mathbb{F}} \cong \text{Ind}_{\mathbb{F}}^F(b|_{\mathbb{F}}) \prec \text{Ind}_{\mathbb{F}}^F(p_{\mathbb{F}}).$$

The action $d = \text{Ind}_{\mathbb{F}}^F(p_{\mathbb{F}})$ is a profinite action, and d is ergodic since $p_{\mathbb{F}}$ is ergodic. As $b \prec d$ there exists an isomorphic copy d_0 of d in $V_1 * V_2$. Then $d_0^\varphi \in U_1 * U_2$, and d_0^φ is ergodic and profinite since d is ergodic and profinite. \square

Note 4.9. The group $\Gamma_1 * \Gamma_2$ never has property (T) when Γ_1 and Γ_2 are non-trivial, so by Corollary 4.7 it would have been enough to show in the above proof that $\Gamma_1 * \Gamma_2$ has MD, and then EMD would follow.

THEOREM 4.10. *The following are equivalent:*

- (1) MD and EMD are equivalent properties for any countably infinite group Γ ;
- (2) EMD passes to subgroups;
- (3) MD is incompatible with property (T).

Proof. (1) \Rightarrow (2): property MD passes to subgroups, so if MD and EMD are equivalent, then EMD passes to subgroups. (2) \Rightarrow (1): if Γ is a countable group with MD then $\Gamma * \Gamma$ has EMD, so if EMD passes to subgroups then Γ actually has EMD. (1) \Rightarrow (3): EMD is incompatible with property (T) since if Γ is an infinite residually finite group with property (T) then p_Γ is strongly ergodic so that $\iota \not\prec p_\Gamma$. Thus, if MD and EMD are equivalent then MD is also incompatible with property (T). (3) \Rightarrow (1): this follows immediately from Corollary 4.7. \square

Note also that Theorem 1.2 gives the following proposition.

PROPOSITION 4.11. *MD is incompatible with $((\tau)$ and $\neg(T))$. That is, if a group Γ has both MD and property (τ) , then Γ actually has property (T).*

Proof. If Γ has MD then by Corollary 4.7, p_Γ is universal for ergodic actions, so if Γ does not have (T) then there exists an ergodic \mathbf{a} with $\iota < \mathbf{a}$. This implies $\iota < p_\Gamma$ so that Γ does not have property (τ) . \square

5. Weak equivalence and invariant random subgroups

5.1. Invariant random subgroups. We let $\text{Sub}(\Gamma)$ denote the set of all subgroups of Γ . This is a compact subset of 2^Γ with the product topology, and is invariant under the left conjugation action of Γ , which is continuous, and which we denote by c , that is, $\gamma^c \cdot H = \gamma H \gamma^{-1}$. We will always view Γ as acting on $\text{Sub}(\Gamma)$ by conjugation, though the underlying measure on $\text{Sub}(\Gamma)$ will vary. By an invariant random subgroup of Γ we mean a conjugation-invariant Borel probability measure θ on $\text{Sub}(\Gamma)$. Invariant random subgroups are studied in [AGV12] as a stochastic generalization of normal subgroups. See also [AE11, Bo10, Ve11]. We let $\text{IRS}(\Gamma)$ denote the space of all invariant random subgroups of Γ . When $\theta \in \text{IRS}(\Gamma)$ we will let θ denote the measure preserving action $\Gamma \curvearrowright^c (\text{Sub}(\Gamma), \theta)$. For a measure preserving action $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ we let $\text{type}(\mathbf{a})$ denote the type of \mathbf{a} , which is defined to be the measure $(\text{stab}_a)_* \mu$ on $\text{Sub}(\Gamma)$, where $\text{stab}_a : X \rightarrow \text{Sub}(\Gamma)$ is the stabilizer map $x \mapsto \text{stab}_a(x) = \Gamma_x = \{\gamma \in \Gamma : \gamma^a x = x\} \in \text{Sub}(\Gamma)$. It is clear that $\text{type}(\mathbf{a})$ is always an IRS of Γ . Types are studied in [AE11] in order to examine freeness properties of measure preserving actions.

5.2. The compact space of weak equivalence classes. Abért and Elek [AE11] define a compact Polish topology on the set of weak equivalence classes of measure preserving actions of Γ . We define this topology below and provide a variation of their proof showing that it is a compact Polish topology.

For this subsection we fix a standard probability space (X, μ) and a compact zero-dimensional Polish space K homeomorphic to Cantor space $2^\mathbb{N}$. We let $\mathcal{K} = \mathcal{K}(M_s(K^\Gamma))$ denote the space of all non-empty compact subsets of $M_s(K^\Gamma)$, equipped with the Vietoris topology τ_V which makes \mathcal{K} into a compact Polish space. Since $M_s(K^\Gamma)$ is a compact metric space, convergence in this topology may be described as follows. A sequence $L_n \in \mathcal{K}$, $n \in \mathbb{N}$, converges if and only if the sets

$$\begin{aligned} \overline{\text{Tlim}}_n L_n &= \{\lambda \in M_s(K^\Gamma) : \exists(\lambda_n) [\forall n \lambda_n \in L_n, \text{ and } \lambda_n \rightarrow \lambda]\} \\ \underline{\text{Tlim}}_n L_n &= \{\lambda \in M_s(K^\Gamma) : \exists(\lambda_n) [\forall n \lambda_n \in L_n, \\ &\quad \text{and for some subsequence } (\lambda_{n_k}), \lambda_{n_k} \rightarrow \lambda]\} \end{aligned}$$

are equal, in which case their common value is the limit of the sequence L_n (see, for example, [Ke95, 4.F]).

Let $\Phi : A(\Gamma, X, \mu) \rightarrow \mathcal{K}$ be the map

$$\Phi(\mathbf{a}) = \overline{E(\mathbf{a}, K)}.$$

By Proposition 3.6, $\Phi(\mathbf{a}) = \Phi(\mathbf{b})$ if and only if $\mathbf{a} \sim \mathbf{b}$. We now have the following theorem.

THEOREM 5.1. *The image of Φ in \mathcal{K} is a closed, hence compact subset of (\mathcal{K}, τ_V) .*

Proof. Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \dots$ be a sequence in $A(\Gamma, X, \mu)$ and suppose that $\Phi(\mathbf{a}_n)$ converges in (\mathcal{K}, τ_V) to the compact set $L \in \mathcal{K}$. We will show that there exists $\mathbf{a}_\infty \in A(\Gamma, X, \mu)$ such that $\Phi(\mathbf{a}_\infty) = L$. Since $E(\mathbf{a}_n, K)$ is dense in $\Phi(\mathbf{a}_n)$ we may write L as

$$\begin{aligned} L &= \{\lambda \in M_s(K^\Gamma) : \exists(\lambda_n) [\forall n \lambda_n \in E(\mathbf{a}_n, K), \text{ and } \lambda_n \rightarrow \lambda]\} \\ &= \{\lambda \in M_s(K^\Gamma) : \exists(\lambda_n) [\forall n \lambda_n \in E(\mathbf{a}_n, K), \\ &\quad \text{and for some subsequence } (\lambda_{n_k}), \lambda_{n_k} \rightarrow \lambda]\}. \end{aligned}$$

Fix a non-principal ultrafilter \mathcal{U} on \mathbb{N} , let $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$ be the ultrapower of the measure space (X, μ) , and let $\mathbf{a}_{\mathcal{U}} = \Gamma \curvearrowright^{\mathbf{a}_{\mathcal{U}}} (X_{\mathcal{U}}, \mu_{\mathcal{U}})$ denote the ultraproduct $\prod_n \mathbf{a}_n / \mathcal{U}$ of the sequence $\{\mathbf{a}_n\}_{n \in \mathbb{N}}$.

CLAIM. $L = E(\mathbf{a}_{\mathcal{U}}, K)$.

Proof of claim. Let $\lambda \in L$ and let $\lambda_n \in E(\mathbf{a}_n, K)$, $n \in \mathbb{N}$, with $\lambda_n \rightarrow \lambda$. For each n there exists $\phi_n : X \rightarrow K$ such that $\lambda_n = (\Phi^{\phi_n, \mathbf{a}_n})_* \mu$. Let $\phi : X_{\mathcal{U}} \rightarrow K$ be the ultralimit of the functions ϕ_n . By Proposition 3.11(2),

$$(\Phi^{\phi, \mathbf{a}_{\mathcal{U}}})_* \mu_{\mathcal{U}} = \lim_{n \rightarrow \mathcal{U}} (\Phi^{\phi_n, \mathbf{a}_n})_* \mu = \lim_{n \rightarrow \mathcal{U}} \lambda_n = \lambda.$$

This shows that $\lambda \in E(\mathbf{a}_{\mathcal{U}}, K)$, and thus $L \subseteq E(\mathbf{a}_{\mathcal{U}}, K)$.

Conversely, let $\lambda \in E(\mathbf{a}_{\mathcal{U}}, K)$, say $\lambda = (\Phi^{\psi, \mathbf{a}_{\mathcal{U}}})_* \mu_{\mathcal{U}}$ for some $\mathbf{B}_{\mathcal{U}}$ -measurable $\psi : X_{\mathcal{U}} \rightarrow K$. By Proposition 3.11(3), we may find a sequence $\phi_n : X \rightarrow K$, $n \in \mathbb{N}$, of Borel functions such that, letting ϕ denote the ultralimit of the ϕ_n , $\mu_{\mathcal{U}}$ -almost everywhere $\psi([x_n]) = \phi([x_n])$. Let $\Phi_n = \Phi^{\phi_n, \mathbf{a}_n}$, let $\Phi = \Phi^{\phi, \mathbf{a}_{\mathcal{U}}}$, and let $\lambda_n = (\Phi_n)_* \mu \in E(\mathbf{a}_n, K)$. Then $\Phi^{\psi, \mathbf{a}_{\mathcal{U}}}([x_n]) = \Phi([x_n])$ almost everywhere, so by Proposition 3.11(2) we have $\lambda = (\Phi^{\psi, \mathbf{a}_{\mathcal{U}}})_* \mu_{\mathcal{U}} = \Phi_* \mu_{\mathcal{U}} = \lim_{n \rightarrow \mathcal{U}} \lambda_n$ so there exists a subsequence $n_0 < n_1 < \dots$ such that $\lambda_{n_k} \rightarrow \lambda$. Hence $\lambda \in L$ and so $E(\mathbf{a}_{\mathcal{U}}, K) \subseteq L$. \square

Let $D \subseteq L$ be a countable dense subset of $L = E(\mathbf{a}_{\mathcal{U}}, K)$. For each $\lambda \in D$ we choose some $\mathbf{B}_{\mathcal{U}}$ -measurable $\phi_\lambda : X_{\mathcal{U}} \rightarrow K$ with $(\Phi^{\phi_\lambda, \mathbf{a}_{\mathcal{U}}})_* \mu_{\mathcal{U}} = \lambda$, and we also choose a sequence $\phi_{\lambda, m} : X_{\mathcal{U}} \rightarrow K$, $m \in \mathbb{N}$, of functions converging in measure to ϕ_λ , such that each $\phi_{\lambda, m}$ is constant on some $\mathbf{B}_{\mathcal{U}}$ -measurable finite partition $\mathcal{P}^{(\lambda, m)}$ of $X_{\mathcal{U}}$. By Theorem A.3 there exists a countably generated standard factor \mathbf{M} of $\text{MALG}_{\mu_{\mathcal{U}}}$ containing $\bigcup_{\lambda \in D} \bigcup_{m \in \mathbb{N}} \mathcal{P}^{(\lambda, m)}$ that is isomorphic to MALG_μ . Let \mathbf{a}_∞ be an action on (X, μ) corresponding to a point realization of the action of Γ on \mathbf{M} by measure algebra automorphisms. It is clear that $E(\mathbf{a}_\infty, K) \subseteq E(\mathbf{a}_{\mathcal{U}}, K) = L$. We show that $D \subseteq \overline{E(\mathbf{a}_\infty, K)}$. Given $\lambda \in D$, each of the functions $\phi_{\lambda, m}$ is \mathbf{M} -measurable, so $(\Phi^{\phi_{\lambda, m}, \mathbf{a}_{\mathcal{U}}})_* \mu_{\mathcal{U}} \in E(\mathbf{a}_\infty, K)$ for all m . Since $\phi_{\lambda, m} \rightarrow \phi_\lambda$ in measure it follows that $(\Phi^{\phi_{\lambda, m}, \mathbf{a}_{\mathcal{U}}})_* \mu_{\mathcal{U}} \rightarrow \lambda$, and thus $\lambda \in \overline{E(\mathbf{a}_\infty, K)}$. Thus $L = \overline{E(\mathbf{a}_\infty, K)}$. \square

For $\mathbf{a} \in A(\Gamma, X, \mu)$ let $[\mathbf{a}] \subseteq A(\Gamma, X, \mu)$ denote the weak equivalence class of \mathbf{a} in $A(\Gamma, X, \mu)$. Let $A_\sim(\Gamma, X, \mu) = \{[\mathbf{a}] : \mathbf{a} \in A(\Gamma, X, \mu)\}$ be the set of all weak equivalence classes of elements of $A(\Gamma, X, \mu)$, and let τ denote the topology on $A_\sim(\Gamma, X, \mu)$ obtained by identifying $A_\sim(\Gamma, X, \mu)$ with a closed subset of (\mathcal{K}, τ_V) via Φ . This makes $A_\sim(\Gamma, X, \mu)$ into a compact metrizable space.

THEOREM 5.2.

- (1) [AE11] The type, $\text{type}(\mathbf{a})$, of a measure preserving action is an invariant of weak equivalence.
- (2) The map $[\mathbf{a}] \mapsto \text{type}(\mathbf{a})$ is a continuous map from the space $(A_{\sim}(\Gamma, X, \mu), \tau)$ of weak equivalence classes of measure preserving actions of Γ to the space $\text{IRS}(\Gamma)$ of invariant random subgroups of Γ equipped with the weak* topology.

Proof. Let $\mathbf{b}_n \in A(\Gamma, X, \mu)$, $n \in \mathbb{N}$, and suppose that $[\mathbf{b}_n] \rightarrow [\mathbf{b}]$ in τ , that is, $\overline{E(\mathbf{b}_n, K)} \rightarrow \overline{E(\mathbf{b}, K)}$ in τ_V . In light of Proposition A.4, both (1) and (2) will follow once we show that $\text{type}(\mathbf{a}_n) \rightarrow \text{type}(\mathbf{a})$ for all $\mathbf{a}_n \in [\mathbf{b}_n]$ and $\mathbf{a} \in [\mathbf{b}]$. Let $\theta_n = \text{type}(\mathbf{a}_n)$ and let $\theta = \text{type}(\mathbf{a})$. Let $F, G \subseteq \Gamma$ be finite. We define $N_F = \{H \in \text{Sub}(\Gamma) : F \cap H = \emptyset\}$, $N_{F,G} = \{H \in \text{Sub}(\Gamma) : F \cap H = \emptyset \text{ and } G \subseteq H\}$ and

$$\begin{aligned} A_F^n &= \bigcap_{\gamma \in F} \text{supp}(\gamma^{a_n}), & A_{F,G}^n &= \bigcap_{\gamma \in F} \text{supp}(\gamma^{a_n}) \cap \bigcap_{\gamma \in G} X \setminus \text{supp}(\gamma^{a_n}), \\ A_F &= \bigcap_{\gamma \in F} \text{supp}(\gamma^a), & A_{F,G} &= \bigcap_{\gamma \in F} \text{supp}(\gamma^a) \cap \bigcap_{\gamma \in G} X \setminus \text{supp}(\gamma^a). \end{aligned}$$

Then $\theta_n(N_F) = \mu(A_F^n)$, $\theta_n(N_{F,G}) = \mu(A_{F,G}^n)$, $\theta(N_F) = \mu(A_F)$, and $\theta(N_{F,G}) = \mu(A_{F,G})$. We will be done once we show that $\mu(A_{F,G}^n) \rightarrow \mu(A_{F,G})$ for all finite $F, G \subseteq \Gamma$.

We first show that $\mu(A_F^n) \rightarrow \mu(A_F)$ for all finite $F \subseteq \Gamma$.

LEMMA 5.3. $\mu(A_F) \leq \liminf_n \mu(A_F^n)$ for all finite $F \subseteq \Gamma$.

Proof. We may write A_F as a countable disjoint union $A_F = \bigsqcup_{m \geq 0} A_m$, where $\mu(\gamma^a A_m \cap A_m) = 0$ for all $\gamma \in F$ and $m \in \mathbb{N}$. Then for any $\epsilon > 0$ we can find M so large that $\sum_{m \geq M} \mu(A_m) < \epsilon/2|F|$. Since $[\mathbf{a}_n] \rightarrow [\mathbf{a}]$ in τ we have that $E(\mathbf{a}, K) \subseteq \text{TLim}_n \overline{E(\mathbf{a}_n, K)}$ so, by Proposition 3.6, $\mathbf{a} < \{\mathbf{a}_n : n \in I\}$ for any infinite $I \subseteq \mathbb{N}$. Thus there exists N such that for each $n > N$ we can find A_0^n, \dots, A_{M-1}^n such that for all $\gamma \in F \cup \{e\}$ and $i, j < M$ we have

$$|\mu(\gamma^a A_i \cap A_j) - \mu(\gamma^{a_n} A_i^n \cap A_j^n)| < \frac{\epsilon}{2M^2|F|}.$$

Then, fixing n with $n > N$, in particular we have $\mu(\gamma^{a_n} A_i^n \cap A_i^n) < (\epsilon/2M^2|F|)$ and $|\mu(A_i) - \mu(A_i^n)| < (\epsilon/2M^2|F|)$ for all $\gamma \in F$ and $i < M$, and $\mu(A_i^n \cap A_j^n) < (\epsilon/2M^2|F|)$ for all $i, j < M, i \neq j$. Define for $i < M$ the sets

$$B_i^n = A_i^n \setminus \left(\bigcup_{\gamma \in F} \gamma^{a_n} A_i^n \cup \bigcup_{j \neq i} A_j^n \right).$$

Then for $\gamma \in F$, $\gamma^{a_n} B_i^n \cap B_i^n = \emptyset$, and for $i \neq j$, $B_i^n \cap B_j^n = \emptyset$. Thus $\bigsqcup B_i^n \subseteq A_F^n$. Since $\mu(B_i^n) \geq \mu(A_i^n) - ((M-1) + |F|)(\epsilon/2M^2|F|) > \mu(A_i) - \epsilon/2M$ it follows that $\mu(A_F^n) \geq \sum_{i < M} \mu(B_i^n) > (\sum_{i < M} \mu(A_i)) - \epsilon/2 > \mu(A_F) - \epsilon$. Since this holds for all $n > N$ and since $\epsilon > 0$ was arbitrary we are done. \square

LEMMA 5.4. $\limsup_n \mu(A_F^n) \leq \mu(A_F)$ for all finite $F \subseteq \Gamma$.

Proof. We may write each A_F^n as a countable disjoint union $A_F^n = \bigsqcup_{m=0}^\infty A_m^n$, where for all $n, m \in \mathbb{N}$, $\gamma^{a_n} \cdot A_m^n \cap A_m^n = \emptyset$. We also define $A_{-1}^n = X \setminus A_F^n$. Let $B_{-1}, B_0, B_1, B_2, \dots$ be a sequence of disjoint non-empty clopen subsets of K , let $k_m \in B_m$, and define $\phi_n : X \rightarrow K$ by $\phi_n(x) = k_m$ for $x \in A_m^n$. The set

$$\begin{aligned} B_F &= \{f \in K^\Gamma : (\forall m \geq -1)[f(e) \in B_m \Rightarrow (\forall \gamma \in F)(f(\gamma) \notin B_m)]\} \\ &= K^\Gamma \setminus \bigcup_{m \geq -1} \left(\pi_e^{-1}(B_m) \cap \bigcup_{\gamma \in F} \pi_\gamma^{-1}(B_m) \right) \end{aligned}$$

is closed and contained in the open set $U_F = \{f : \forall \gamma \in F \ f(\gamma) \neq f(e)\}$. Fixing n , for each $m \geq 0$ we have that

$$(\Phi^{\phi_n, a_n})^{-1} \left(\pi_e^{-1}(B_m) \cap \bigcup_{\gamma \in F} \pi_\gamma^{-1}(B_m) \right) = A_m^n \cap \bigcup_{\gamma \in F} \gamma^{a_n} A_F^n = \emptyset,$$

while for $m = -1$ we have that $(\Phi^{\phi_n, a_n})^{-1}(\pi_e^{-1}(B_{-1}) \cap \bigcup_{\gamma \in F} \pi_\gamma^{-1}(B_{-1})) = A_{-1}^n$ since $A_{-1}^n \subseteq \bigcup_{\gamma \in F} \gamma^{a_n} A_{-1}^n$. It follows that $(\Phi^{\phi_n, a_n})^{-1}(B_F) = A_F^n$. Let $\lambda_n = (\Phi^{\phi_n, a_n})_* \mu \in E(a_n, K)$. Take any convergent subsequence $\{\lambda_{n_k}\}$, and let $\lambda = \lim_k \lambda_{n_k}$. Since $\overline{E(a_n, K)} \rightarrow \overline{E(a, K)}$ we have that $\lambda \in \overline{E(a, K)}$, so let $\rho_n = (\Phi^{\psi_n, a})_* \mu \in E(a, K)$ be such that $\rho_n \rightarrow \lambda$. We now have

$$\begin{aligned} \limsup_k \mu(A_F^{n_k}) &= \limsup_k \lambda_{n_k}(B_F) \leq \lambda(B_F) \leq \lambda(U_F) \\ &\leq \liminf_n \rho_n(U_F) = \liminf_n \mu(\{x : \forall \gamma \in F \ \psi_n((\gamma^{-1})^a x) \neq \psi_n(x)\}) \\ &\leq \mu(A_F). \end{aligned}$$

Since the convergent subsequence (λ_{n_k}) was arbitrary we conclude that $\limsup_n \mu(A_F^n) \leq \mu(A_F)$. \square

It follows from the above two lemmas that $\mu(A_F) = \lim_n \mu(A_F^n)$ for all finite $F \subseteq \Gamma$. Now let $F, G \subseteq \Gamma$ be finite and note that $A_F^n = A_{F,G}^n \sqcup \bigcup_{\gamma \in G} A_{F \cup \{\gamma\}}^n$ and $A_F = A_{F,G} \sqcup \bigcup_{\gamma \in G} A_{F \cup \{\gamma\}}$. We have just shown that $\mu(A_F) = \lim_n \mu(A_F^n)$. By the inclusion-exclusion principle we have

$$\mu\left(\bigcup_{\gamma \in G} A_{F \cup \{\gamma\}}^n\right) = \sum_{k=1}^{|G|} (-1)^{k-1} \sum_{\{J \subseteq G : |J|=k\}} \mu(A_{F \cup J}^n),$$

and since $\mu(A_{F \cup J}^n) \rightarrow \mu(A_{F \cup J})$ for each $J \subseteq G$ it follows after another application of inclusion-exclusion that $\mu(\bigcup_{\gamma \in G} A_{F \cup \{\gamma\}}^n) \rightarrow \mu(\bigcup_{\gamma \in G} A_{F \cup \{\gamma\}})$. Thus $\mu(A_{F,G}^n) \rightarrow \mu(A_{F,G})$. \square

COROLLARY 5.5. [AE11] For each $\theta \in \text{IRS}(\Gamma)$, $\{[a] : \text{type}(a) = \theta\} \subseteq A_\sim(\Gamma, X, \mu)$ is compact in τ . In particular, $\{[a] : [a] \text{ is free}\}$ is compact in τ .

Remark 5.6. The technique used in the proof of Theorem 5.2 can be used to show that combinatorial invariants of measure preserving actions such as independence number (see [CK13, CKT-D13]) are continuous functions on $(A_\sim(\Gamma, X, \mu), \tau)$.

THEOREM 5.7. Let Γ be a countable group.

- (1) The map $(A(\Gamma, X, \mu), w) \rightarrow (A_\sim(\Gamma, X, \mu), \tau)$, $\mathbf{a} \mapsto [\mathbf{a}]$ is Baire class 1. In particular, for each $\theta \in \text{IRS}(\Gamma)$ the space $\{\mathbf{a} \in A(\Gamma, X, \mu) : \text{type}(\mathbf{a}) = \theta\}$ is a G_δ hence Polish subspace of $(A(\Gamma, X, \mu), w)$.
- (2) The topology τ is a refinement of the quotient topology on $A_\sim(\Gamma, X, \mu)$ induced by w . If (X, μ) is not a discrete space and $\Gamma \neq \{e\}$ then the τ topology is strictly finer than the quotient topology.

Proof. We begin with (1). For this we show that $\mathbf{a} \mapsto \overline{E(\mathbf{a}, K)} \in \mathcal{K}$ is Baire class 1. We observe that $\{\mathbf{a} : \overline{E(\mathbf{a}, K)} \subseteq C\}$ is closed in $(A(\Gamma, X, \mu), w)$ whenever $C \subseteq M_s(K^\Gamma)$ is closed. This is because if $\mathbf{a}_n \in A(\Gamma, X, \mu)$, $n \in \mathbb{N}$, is such that $\overline{E(\mathbf{a}_n, K)} \subseteq C$ and $\mathbf{a}_n \rightarrow \mathbf{a} \in A(\Gamma, X, \mu)$ in the weak topology then $E(\mathbf{a}, K) \subseteq \bigcup_n \overline{E(\mathbf{a}_n, K)} \subseteq C$.

The topology τ_V on \mathcal{K} is generated by the sets $\{L : L \subseteq U\}$ and $\{L : L \cap U \neq \emptyset\}$, where U ranges over all open subsets of $M_s(K^\Gamma)$. For any open $U \subseteq M_s(K^\Gamma)$ the above observation shows that $\{\mathbf{a} : \overline{E(\mathbf{a}, K)} \cap U \neq \emptyset\}$ is open, and if we write $U = \bigcup_n C_n$, where each C_n is closed and $C_n \subseteq \text{int}(C_{n+1})$, then $\{\mathbf{a} : \overline{E(\mathbf{a}, K)} \subseteq U\} = \bigcup_n \{\mathbf{a} : \overline{E(\mathbf{a}, K)} \subseteq C_n\}$, which is F_σ .

For the first part of (2) we note that the following are equivalent for a subset \mathcal{B} of $A(\Gamma, X, \mu)$:

- (i) \mathcal{B} is weakly closed and for all $\mathbf{a}, \mathbf{b} \in A(\Gamma, X, \mu)$, $\mathbf{a} \in \mathcal{B}$ and $\mathbf{b} \sim \mathbf{a}$ implies $\mathbf{b} \in \mathcal{B}$;
- (ii) \mathcal{B} is weakly closed and for all $\mathbf{a}, \mathbf{b} \in A(\Gamma, X, \mu)$, $\mathbf{a} \in \mathcal{B}$ and $\mathbf{b} \cong \mathbf{a}$ implies $\mathbf{b} \in \mathcal{B}$;
- (iii) for all $\mathbf{a} \in A(\Gamma, X, \mu)$, $\mathbf{a} \prec \mathcal{B}$ implies $\mathbf{a} \in \mathcal{B}$.

The implication (i) \Rightarrow (ii) is trivial, (ii) \Rightarrow (iii) follows from Proposition 3.6, and (iii) \Rightarrow (i) follows from the fact that if $\mathbf{a}_n \rightarrow \mathbf{a}$ in $A(\Gamma, X, \mu)$ then $\mathbf{a} \prec \{\mathbf{a}_n\}_{n \in \mathbb{N}}$. To show the first part of (2) it suffices to show that if \mathcal{B} satisfies the above equivalent properties, then $\mathcal{B}_\sim = \{[\mathbf{a}] : \mathbf{a} \in \mathcal{B}\}$ is closed in τ . Let $L = \bigcup_{\mathbf{a} \in \mathcal{B}} \overline{E(\mathbf{a}, K)}$. Then $L \subseteq M_s(K^\Gamma)$ is closed and property (iii) tells us that $\mathcal{B}_\sim = \{[\mathbf{a}] \in A_\sim(\Gamma, X, \mu) : \overline{E(\mathbf{a}, K)} \subseteq L\}$, which is exactly the definition of a basic closed set in τ_V .

Suppose that (X, μ) is not discrete and let $C \subseteq X$ be the continuous part of X so that $\mu(C) > 0$. Then (C, μ_C) is a standard non-atomic probability space so there exists a universal measure preserving action $\mathbf{a} = \Gamma \curvearrowright^a (C, \mu_C)$ weakly containing all other measure preserving actions of Γ . Let b be the action of Γ on (X, μ) whose restriction to C is equal to a and whose restriction to $X \setminus C$ is identity and let $\mathbf{b} = \Gamma \curvearrowright^b (X, \mu)$. As $\iota_{\mu_C} \prec \mathbf{a}$ by Lemma 3.5 there exist isomorphic copies of \mathbf{a} converging to ι_{μ_C} in $A(\Gamma, C, \mu_C)$. This yields isomorphic copies of \mathbf{b} converging to ι_μ in $A(\Gamma, X, \mu)$. Thus $[\iota_\mu]$ is in the closure of $\{[\mathbf{b}]\}$ in the quotient topology, but $[\iota_\mu]$ is not in the τ topology closure of $\{[\mathbf{b}]\}$ since $\Gamma \neq \{e\}$ so that $[\iota_\mu] \neq [\mathbf{b}]$. \square

Remark 5.8. The map $\mathcal{K} \rightarrow \mathcal{K}$ sending $L \mapsto \overline{\text{co}}L$ is continuous in the Vietoris topology τ_V . Indeed, if $L_n \rightarrow L_\infty$ we show that $\overline{\text{Tlim}_n \overline{\text{co}}L_n} \subseteq \overline{\text{co}}L_\infty \subseteq \overline{\text{Tlim}_n \overline{\text{co}}L_n}$. Let $\lambda \in \overline{\text{Tlim}_n \overline{\text{co}}L_n}$ so that there exists $\lambda_{n_k} \in \overline{\text{co}}L_{n_k}$ with $\lambda_{n_k} \rightarrow \lambda$. Then there exist probability measures μ_{n_k} on $M_s(K^\Gamma)$ supported on L_{n_k} with $\lambda_{n_k} = \int_{\rho \in M_s(K^\Gamma)} \rho d\mu_{n_k}$ and (after moving to a subsequence if necessary) we may assume that μ_{n_k} converges to some measure μ on $M_s(K^\Gamma)$. Then $\lambda = \int_{\rho \in M_s(K^\Gamma)} \rho d\mu$. Let $C_0 \supseteq C_1 \supseteq \dots$ be a sequence of closed subsets of $M_s(K^\Gamma)$ with $L_\infty \subseteq \text{int}(C_m)$ for all m and $L_\infty = \bigcap_m C_m$. For each m the

set $\{L \in \mathcal{K} : L \subseteq C_m\}$ is a neighborhood of L_∞ in \mathcal{K} and so contains L_{n_k} for all large enough k . It follows that $\mu(C_m) \geq \liminf_k \mu_{n_k}(C_m) = 1$, and so $\mu(L_\infty) = \lim_m \mu(C_m) = 1$. Since μ is supported on L_∞ and has barycenter λ , it follows that $\lambda \in \overline{\text{co}}L_\infty$. For the second inclusion it is easy to see that $\text{co}L_\infty \subseteq \overline{\text{Tlim}_n \text{co}L_n}$ and since the latter set is closed it follows that $\overline{\text{co}}L_\infty \subseteq \overline{\text{Tlim}_n \text{co}L_n}$.

If now \mathbf{a} is a measure preserving action of Γ and (Y, ν) is non-atomic then \mathbf{a} is stably weakly equivalent to an action on (Y, ν) and we let $[\mathbf{a}]_s = \{\mathbf{b} \in A(\Gamma, Y, \nu) : \mathbf{b} \sim_s \mathbf{a}\}$ denote the stable weak equivalence class of \mathbf{a} in (Y, ν) (see Definition B.1). It follows that the space $A_{\sim_s}(\Gamma, Y, \nu) = \{[\mathbf{a}]_s : \mathbf{a} \text{ is a measure preserving action of } \Gamma\}$ of all stable weak equivalence classes of measure preserving actions of Γ may be viewed as a compact subset of \mathcal{K} via the map $[\mathbf{a}]_s \mapsto \overline{\text{co}}E(\mathbf{a}, K)$. Since $\text{type}(\mathbf{a}) = \text{type}(\iota \times \mathbf{a})$ it follows that $\text{type}(\mathbf{a})$ is an invariant of stable weak equivalence. The map $[\mathbf{a}] \mapsto \text{type}(\mathbf{a})$ then factors through $[\mathbf{a}] \mapsto [\mathbf{a}]_s$, and so Theorem 5.2 also holds for stable weak equivalence.

5.3. Random Bernoulli shifts. Given $\theta \in \text{IRS}(\Gamma)$, one constructs a measure preserving action of Γ of type θ as follows (see [AGV12, Proposition 45]).

Fix a standard probability space (Z, η) and let $Z^{\leq \Gamma} = \bigsqcup_{H \in \text{Sub}(\Gamma)} Z^{H \setminus \Gamma}$. Here, $H \setminus \Gamma$ denotes the collection of right cosets of H in Γ . We define the projection map $Z^{\leq \Gamma} \rightarrow \text{Sub}(\Gamma)$, $f \mapsto H_f \in \text{Sub}(\Gamma)$, where $H_f = H$ when $f \in Z^{H \setminus \Gamma}$. We endow $Z^{\leq \Gamma}$ with the standard Borel structure it inherits as a Borel subset of $Z^\Gamma \times \text{Sub}(\Gamma)$ via the injection $f \mapsto ((\gamma \mapsto f(H_f \gamma)), H_f)$. The image of $Z^{\leq \Gamma}$ under this map is invariant under the product action $\tilde{s} \times c$ of Γ on $Z^\Gamma \times \text{Sub}(\Gamma)$ (where \tilde{s} denotes the shift action of Γ on Z^Γ), and we let s denote the corresponding action of Γ on $Z^{\leq \Gamma}$. We have that $H_{\gamma^s f} = \gamma H_f \gamma^{-1}$ for each $\gamma \in \Gamma$ and $f \in Z^{\leq \Gamma}$ and $(\gamma^s f)(\gamma H_f \gamma^{-1} \delta) = f(H_f \gamma^{-1} \delta)$. Let $\eta^{H \setminus \Gamma}$ denote the product measure on $Z^{H \setminus \Gamma} \subseteq Z^{\leq \Gamma}$, and observe that under this action we have $(\gamma^s)_* \eta^{H \setminus \Gamma} = \eta^{(\gamma H \gamma^{-1}) \setminus \Gamma}$. It follows that the measure $\eta^{\theta \setminus \Gamma}$ on $Z^{\leq \Gamma}$ defined by

$$\eta^{\theta \setminus \Gamma} = \int_H \eta^{H \setminus \Gamma} d\theta(H)$$

is invariant under the action of Γ . We let $s_{\theta, \eta}$ denote the measure preserving action $\Gamma \curvearrowright^s (Z^{\leq \Gamma}, \eta^{\theta \setminus \Gamma})$, and we call $s_{\theta, \eta}$ the θ -random Bernoulli shift of Γ over (Z, η) . This action always contains θ as a factor via the ‘projection’ map $f \mapsto H_f$. When η is non-atomic, the stabilizer map $f \mapsto \Gamma_f$ of $s_{\theta, \eta}$ coincides almost everywhere with this projection. Indeed, if η is non-atomic then for $\eta^{\theta \setminus \Gamma}$ -almost every f the function $f : H \setminus \Gamma \rightarrow Z$ is injective. Since every $\gamma \in \Gamma_f$ satisfies $f(H \gamma^{-1}) = f(H)$, the inclusion $\Gamma_f \subseteq H_f$ is immediate for injective f , and as $H_f \subseteq \Gamma_f$ always holds we conclude that $\Gamma_f = H_f$ almost surely. In particular, $\text{type}(s_{\theta, \eta}) = \theta$. We have thus shown the following proposition.

PROPOSITION 5.9. [AGV12, Proposition 45] *Let Γ be a countable group. For every $\theta \in \text{IRS}(\Gamma)$ there exists a measure preserving action of type θ . Namely, the θ -random Bernoulli shift $s_{\theta, \eta}$ over a non-atomic base space (Z, η) has type θ .*

It is clear that an isomorphism $(Z_1, \eta_1) \cong (Z_2, \eta_2)$ of measure spaces induces an isomorphism $s_{\theta, \eta_1} \cong s_{\theta, \eta_2}$. The next proposition characterizes precisely when $\text{type}(s_{\theta, \eta}) = \theta$ for various η . Below, we write $N(H)$ for the normalizer of a subgroup H of Γ .

PROPOSITION 5.10. Let Γ be a countable group, let $\theta \in \text{IRS}(\Gamma)$, and let (Z, η) be a standard probability space.

- (1) If η is non-atomic then $\Gamma_f = H_f$ almost surely.
- (2) If η is a point mass then $\Gamma_f = N(H_f)$ almost everywhere and the map $f \mapsto H_f$ is an isomorphism $s_{\theta, \eta} \cong \theta$ so that $\text{type}(s_{\theta, \eta}) = \text{type}(\theta)$.
- (3) Suppose that η is not a point mass. Then for each infinite index subgroup of $H \leq \Gamma$, $\Gamma_f = H_f$ for $\eta^{H \setminus \Gamma}$ -almost every $f \in Z^{H \setminus \Gamma}$. Thus, if

$$\theta(\{H : [\Gamma : H] < \infty \text{ and } N(H) \neq H\}) = 0$$

then $\Gamma_f = H_f$ almost surely. In particular, if θ concentrates on the infinite index subgroups of Γ then $\Gamma_f = H_f$ almost surely.

- (4) Suppose that η contains atoms. If

$$\theta(\{H : [\Gamma : H] < \infty \text{ and } N(H) \neq H\}) > 0$$

then $\text{type}(s_{\theta, \eta}) \neq \theta$.

In particular, $\text{type}(s_{\theta, \eta}) = \theta$ if and only if $H_f = \Gamma_f$ almost surely.

Proof. We have already shown (1) in Proposition 5.9 and (2) is clear. For (3) fix an infinite index $H \leq \Gamma$ along with some $\gamma \notin H$ and inductively define an infinite sequence $\{\delta_n\}_{n \in \mathbb{N}}$ by taking $\delta_{n+1} \in \Gamma$ to be any element of the complement of $\bigcup_{i \leq n} (H\delta_i \cup H\gamma^{-1}\delta_i \cup (\gamma H\gamma^{-1})\delta_i \cup (\gamma H\gamma^{-1})(\gamma\delta_i))$ (we are using here the fact that the collection $\{H\delta : H \in \text{Sub}(\Gamma), \delta \in \Gamma, \text{ and } [\Gamma : H] = \infty\}$ of all right cosets of infinite index subgroups of Γ generates a proper ideal of Γ (see, for example, [Ke05, Proof of Lemma 4.4])). By construction all of the cosets $H\delta_0, H\gamma^{-1}\delta_0, H\delta_1, H\gamma^{-1}\delta_1, \dots$ are distinct so, letting $A \subseteq Z$ be any set with $0 < \eta(A) < 1$, it follows that

$$\begin{aligned} \eta^{H \setminus \Gamma}(\{f : \gamma \in \Gamma_f\}) &\leq \eta^{H \setminus \Gamma}(\{f : \forall \delta \in \Gamma (f(H\delta) = f(H\gamma^{-1}\delta))\}) \\ &\leq \eta^{H \setminus \Gamma} \left(\bigcap_{n \in \mathbb{N}} \{f : f(H\delta_n), f(H\gamma^{-1}\delta_n) \in A \text{ or } f(H\delta_n), f(H\gamma^{-1}\delta_n) \notin A\} \right) \\ &= \lim_{N \rightarrow \infty} (\eta(A)^2 + (1 - \eta(A))^2)^N = 0. \end{aligned}$$

Thus $\gamma \notin \Gamma_f$ for $\eta^{H \setminus \Gamma}$ -almost every f , and since this is true for each $\gamma \notin H$ we obtain $\Gamma_f \subseteq H$ for $\eta^{H \setminus \Gamma}$ -almost every f .

We now prove (4). Let $\theta_s = \text{type}(s_{\theta, \eta})$. Let $z_0 \in Z$ be an atom for the measure η . The set $A = \{f \in Z^{\leq \Gamma} : [\Gamma : H_f] < \infty, N(H_f) \neq H_f \text{ and } \forall \gamma \in \Gamma (f(H_f\gamma) = z_0)\}$ is $\eta^{\theta \setminus \Gamma}$ -non-null and $\Gamma_f = N(H_f) \neq H_f$ for each $f \in A$. Thus $[\Gamma : \Gamma_f] = [\Gamma : N(H_f)] < [\Gamma : H_f]$ for each $f \in A$. When $f \notin A$ we still have $[\Gamma : \Gamma_f] \leq [\Gamma : H_f]$. It follows that

$$\begin{aligned} \int_H \frac{1}{[\Gamma : H]} d\theta_s &= \int_{f \in A} \frac{1}{[\Gamma : \Gamma_f]} d\eta^{\theta \setminus \Gamma} + \int_{f \notin A} \frac{1}{[\Gamma : \Gamma_f]} d\eta^{\theta \setminus \Gamma} \\ &> \int_{f \in A} \frac{1}{[\Gamma : H_f]} d\eta^{\theta \setminus \Gamma} + \int_{f \notin A} \frac{1}{[\Gamma : H_f]} d\eta^{\theta \setminus \Gamma} \\ &= \int_f \frac{1}{[\Gamma : H_f]} d\eta^{\theta \setminus \Gamma} = \int_H \frac{1}{[\Gamma : H]} d\theta \end{aligned}$$

and so $\theta_s \neq \theta$, which finishes (4).

It is clear that $\Gamma_f = H_f$ almost everywhere implies $\text{type}(s_{\theta,\eta}) = \theta$. Suppose now that $\Gamma_f \neq H_f$ for a non-null set of $f \in Z^{\leq \Gamma}$. Then (1) implies that η contains atoms and (3) implies that the set $J = \{f \in Z^{\leq \Gamma} : [\Gamma : H_f] < \infty \text{ and } \Gamma_f \neq H_f\}$ is non-null. The inclusions $H_f \subseteq \Gamma_f \subseteq N(H_f)$ hold for all $f \in Z^{\leq \Gamma}$ and so

$$\theta(\{H : [\Gamma : H] < \infty \text{ and } N(H) \neq H\}) \geq \eta^{\theta \setminus \Gamma}(J) > 0.$$

Part (4) now implies that $\text{type}(s_{\theta,\eta}) \neq \theta$. □

THEOREM 5.11. *Let Γ be a countable group, let $\theta \in \text{IRS}(\Gamma)$, and let $s_{\theta,\eta}$ be the θ -random Bernoulli shift over the standard measure space (Z, η) . Let $p : Z^{\leq \Gamma} \rightarrow \text{Sub}(\Gamma)$ denote the projection $p(f) = H_f$ factoring $s_{\theta,\eta}$ onto θ . Assume that η is not a point mass. Then the following are equivalent.*

- (1) θ concentrates on the infinite index subgroups of Γ .
- (2) The extension $p : s_{\theta,\eta} \rightarrow \theta$ is ergodic.
- (3) The extension $p : s_{\theta,\eta} \rightarrow \theta$ is weak mixing.

In particular, if θ is infinite index then $s_{\theta,\eta}$ is ergodic if and only if θ is ergodic.

Proof. (3) \Rightarrow (2) is trivial. For (2) \Rightarrow (1), suppose that $\theta(C) > 0$, where $C = \{H : [\Gamma : H] < \infty\}$ and let $A \subseteq Z$ be any measurable set with $0 < \eta(A) < 1$. Then the set $B = \{f \in Z^{\leq \Gamma} : H_f \in C \text{ and } \text{ran}(f) \subseteq A\}$ is a non-trivial invariant set that is not p -measurable.

To show that (1) \Rightarrow (3), we must show that the extension $\tilde{p} : s_{\theta,\eta} \otimes_{\theta} s_{\theta,\eta} \rightarrow \theta$ is ergodic, where

$$s_{\theta,\eta} \otimes_{\theta} s_{\theta,\eta} = \Gamma \curvearrowright^{s \times s} \left(Z^{\leq \Gamma} \times Z^{\leq \Gamma}, \int_H \eta^{H \setminus \Gamma} \times \eta^{H \setminus \Gamma} d\theta \right)$$

and $\tilde{p}(f, g) = p(f)$. Let $(Y, \nu) = (Z \times Z, \eta \times \eta)$. Then we have the natural isomorphism $\varphi : s_{\theta,\eta} \otimes_{\theta} s_{\theta,\eta} \cong s_{\theta,\nu}$ such that $\tilde{p}(f, g) = p \circ \varphi(f, g)$ almost surely, so it suffices to show that the extension $p : s_{\theta,\nu} \rightarrow \theta$ is ergodic. If $\theta = \int_{w \in W} \theta(w) d\rho(w)$ is the ergodic decomposition of θ then $s_{\theta,\nu}$ decomposes as $s_{\theta,\nu} = \int_{w \in W} s_{\theta_w,\nu} d\rho(w)$ and $p : Y^{\leq \Gamma} \rightarrow \text{Sub}(\Gamma)$ factors $s_{\theta_w,\nu}$ onto θ_w almost surely. We may therefore assume that θ is ergodic toward the goal of showing that $s_{\theta,\nu}$ is ergodic as well.

Since θ is ergodic, the index i of $N(H)$ in Γ is constant on a θ -conull set. If $i < \infty$ then the orbit of almost every H is finite and ergodicity implies that there exists an $H_0 \in \text{Sub}(\Gamma)$ such that θ concentrates on the conjugates of H_0 . Then H_0 is an infinite index normal subgroup of $K_0 = N(H_0)$ which implies that the generalized Bernoulli shift action $s = K_0 \curvearrowright^s (Y^{H_0 \setminus \Gamma}, \eta^{H_0 \setminus \Gamma})$ is ergodic (see, for example, [KT09]). Example 5.13 below then shows that $s_{\theta,\nu} \cong \text{Ind}_{K_0}^{\Gamma}(s)$, and so $s_{\theta,\nu}$ is ergodic.

If $i = \infty$ then we proceed as follows. Let $(X, \mu) = (Y^{\leq \Gamma}, \nu^{\leq \Gamma})$ and suppose toward a contradiction that $B \subseteq X$ is invariant and $0 < \mu(B) = r < 1$. The map $H \mapsto \nu^{H \setminus \Gamma}(B)$ is conjugation invariant so ergodicity of θ implies that $\nu^{H \setminus \Gamma}(B) = \mu(B) = r$ almost surely. Let $\epsilon > 0$ be small depending on r . Fix some countable Boolean algebra \mathbf{A}_0 generating $\mathbf{B}(Y)$ and let \mathbf{A} be the countable Boolean algebra of subsets of X generated by $\{\pi_{\gamma}^{-1}(D) : D \in \mathbf{A}_0 \text{ and } \gamma \in \Gamma\}$, where $\pi_{\gamma}(f) = f(H_f \gamma)$ for $f \in X$. Then for every $\epsilon > 0$ there exist $A_1, \dots, A_n \in \mathbf{A}$ and a partition C_0, \dots, C_{n-1} of $\text{Sub}(\Gamma)$ into non-null measurable sets such that $\mu(A \Delta B) < \epsilon^2$, where $A = \bigsqcup_{i < n} (A_i \cap p^{-1}(C_i))$. There exist a finite $F \subseteq \Gamma$ and

a collection $\{D_\delta^{i,j} : \delta \in F, j < n_i, i < n\} \subseteq A_0$ such that $A_i = \bigcup_{0 \leq j < n_i} \bigcap_{\delta \in F} \pi_\delta^{-1}(D_\delta^{i,j})$ for each $i < n$.

LEMMA 5.12. *Let $C \subseteq \text{Sub}(\Gamma)$ be any non-null measurable set. Then for θ -almost every $H \in \text{Sub}(\Gamma)$ there exists $\gamma \in \Gamma$ such that $\{H\alpha\}_{\alpha \in F} \cap \{H\gamma^{-1}\delta\}_{\delta \in F} = \emptyset$ and $\gamma H \gamma^{-1} \in C$.*

Proof. Since θ is ergodic and $[\Gamma : N(H)] = \infty$ almost surely, the intersection C^H , of C with the orbit of H , is almost surely infinite. Fix such an H with both $[\Gamma : N(H)] = \infty$ and C^H infinite. Since the set $FF^{-1} \cdot H = \{\delta\alpha^{-1}H\alpha\delta^{-1} : \alpha, \delta \in F\}$ is finite there exists $\gamma \in \Gamma$ with $\gamma H \gamma^{-1} \in C^H \setminus (FF^{-1} \cdot H)$. This γ works: $\gamma H \gamma^{-1} \notin FF^{-1} \cdot H$ is equivalent to $\gamma \notin \bigcup_{\alpha, \delta \in F} \delta\alpha^{-1}N(H)$, so if $\alpha, \delta \in F$ then $\gamma \notin \delta\alpha^{-1}N(H)$ and thus $H\alpha \neq H\gamma^{-1}\delta$. \square

Using this lemma and measure-theoretic exhaustion we may find a Borel function $\text{Sub}(\Gamma) \rightarrow \Gamma, H \mapsto \gamma_H$, with $\{H\alpha\}_{\alpha \in F} \cap \{H\gamma_H^{-1}\delta\}_{\delta \in F} = \emptyset$ and $\gamma_H H \gamma_H^{-1} \in C_i$ for almost every $H \in C_i$, and such that the function $\psi : \text{Sub}(\Gamma) \rightarrow \text{Sub}(\Gamma), H \mapsto \gamma_H H \gamma_H^{-1}$, is injective on a conull set. In particular, ψ is measure preserving. Let $\varphi : X \rightarrow X$ be given by $\varphi(f) = (\gamma_{H_f})^s \cdot f$ so that φ is also injective on a conull set and measure preserving.

For $H \leq \Gamma$ and $D \subseteq X$ let $D_H = D \cap Y^{H \setminus \Gamma}$. Then for each $i < n$ and almost every $H \in C_i$ we have $\gamma_H H \gamma_H^{-1} \in C_i$ and

$$\varphi(A)_{\gamma_H H \gamma_H^{-1}} = (\gamma_H)^s \cdot ((A_i)_H) = \bigcup_{j < n_i} \bigcap_{\alpha \in F} \{f \in Y^{\gamma_H H \gamma_H^{-1} \setminus \Gamma} : f(\gamma_H H \gamma_H^{-1} \gamma_H \alpha) \in D_\alpha^{i,j}\},$$

$$A_{\gamma_H H \gamma_H^{-1}} = (A_i)_{\gamma_H H \gamma_H^{-1}} = \bigcup_{j < n_i} \bigcap_{\delta \in F} \{f \in Y^{\gamma_H H \gamma_H^{-1} \setminus \Gamma} : f(\gamma_H H \gamma_H^{-1} \delta) \in D_\delta^{i,j}\}.$$

By our choice of γ_H the sets $\{\gamma_H H \gamma_H^{-1} \gamma_H \alpha\}_{\alpha \in F}$ and $\{\gamma_H H \gamma_H^{-1} \delta\}_{\delta \in F}$ are almost surely disjoint and it follows that the sets A and $\varphi(A)$ are $v^{\gamma_H H \gamma_H^{-1} \setminus \Gamma}$ -independent almost surely. Since $H \mapsto \gamma_H H \gamma_H^{-1}$ is a measure preserving injection it follows that A and $\varphi(A)$ are $v^{H \setminus \Gamma}$ -independent almost surely.

We have

$$\epsilon^2 > \mu(A \Delta B) = \int_H v^{H \setminus \Gamma}(A \Delta B) d\theta \geq \int_H |v^{H \setminus \Gamma}(A) - r| d\theta$$

so that $\theta(\{|H : |v^{H \setminus \Gamma}(A) - r| \leq \epsilon\}) \geq 1 - \epsilon$ and since $\mu(A \Delta B) = \mu(\varphi(A) \Delta B)$ we also have $\theta(\{|H : |v^{H \setminus \Gamma}(\varphi(A)) - r| \leq \epsilon\}) \geq 1 - \epsilon$. Then

$$\begin{aligned} r = \mu(B) &\leq \mu(A \Delta B) + \mu(\varphi(A) \Delta B) + \mu(A \cap \varphi(A)) \\ &< 2\epsilon^2 + \int_H v^{H \setminus \Gamma}(A) v^{H \setminus \Gamma}(\varphi(A)) d\theta \leq 2\epsilon^2 + 2\epsilon + (r + \epsilon)^2 \rightarrow_{\epsilon \rightarrow 0} r^2. \end{aligned}$$

This is a contradiction for small enough ϵ since $0 < r < 1$. \square

Example 5.13. The simplest example of an ergodic $\theta \in \text{IRS}(\Gamma)$ is a point mass $\theta = \delta_N$ on some normal subgroup $N \triangleleft \Gamma$. The corresponding random Bernoulli shift $s_{\delta_N, \eta}$ is isomorphic to the usual generalized shift action of Γ on $(Z^{\Gamma/N}, \eta^{\Gamma/N})$.

Almost as simple is when $\theta \in \text{IRS}(\Gamma)$ has the form $\theta = (1/n) \sum_{i=0}^{n-1} \delta_{\gamma_i H \gamma_i^{-1}}$, where $H \leq \Gamma$ is a subgroup with finitely many conjugates $\gamma_0 H \gamma_0^{-1}, \gamma_1 H \gamma_1^{-1}, \gamma_2 H \gamma_2^{-1}, \dots, \gamma_{n-1} H \gamma_{n-1}^{-1}$. Clearly θ is ergodic. In this case the random Bernoulli shift $s_{\theta, \eta}$ may be described

as follows. The set $T = \{\gamma_i\}_{i < n}$ is a transversal for the left cosets of the normalizer $K = N(H)$ of H in Γ , and the natural action of Γ on T given by $\gamma \cdot t \in \gamma t K \cap T$ for $\gamma \in \Gamma$ and $t \in T$ preserves normalized counting measure ν_T on T . Since H is normal in K , the restriction to K of the action s leaves $Z^{H \setminus \Gamma}$ invariant and preserves the measure $\eta^{H \setminus \Gamma}$ so that $s = K \curvearrowright^s (Z^{H \setminus \Gamma}, \eta^{H \setminus \Gamma})$ becomes the usual generalized Bernoulli shift. We let \mathbf{b} denote the induced action $\mathbf{b} = \text{Ind}_K^\Gamma(s)$, which is the measure preserving action $\Gamma \curvearrowright^{\mathbf{b}} (Z^{H \setminus \Gamma} \times T, \eta^{H \setminus \Gamma} \times \nu_T)$ given by

$$\gamma^{\mathbf{b}}(f, t) = (\rho(\gamma, t)^s f, \gamma \cdot t),$$

where $\rho : \Gamma \times T \rightarrow K$ is the cocycle given by $\rho(\gamma, t) = (\gamma \cdot t)^{-1} \gamma t$. The map $\pi : Z^{H \setminus \Gamma} \times T \rightarrow Z^{\Gamma \setminus \Gamma}$ given by $\pi(f, t) = t^s f \in Z^{t H t^{-1} \setminus \Gamma}$ is an isomorphism of \mathbf{b} with $s_{\theta, \eta}$. Indeed, π is equivariant since

$$\pi(\gamma^{\mathbf{b}}(f, t)) = \pi(\rho(\gamma, t)^s f, \gamma \cdot t) = (\gamma \cdot t)^s \rho(\gamma, t)^s f = (\gamma t)^s f = \gamma^s t^s f = \gamma^s \pi(f, t),$$

and π is measure preserving since

$$\pi_*(\eta^{H \setminus \Gamma} \times \nu_T) = \frac{1}{n} \sum_{t \in T} \pi_*(\eta^{H \setminus \Gamma} \times \delta_t) = \frac{1}{n} \sum_{t \in T} \eta^{t H t^{-1} \setminus \Gamma} = \eta^{\theta \setminus \Gamma}.$$

It is also clear that π is injective since $t \mapsto t H t^{-1}$ is a bijection of T with the conjugates of H .

5.4. A sufficient condition for weak containment.

Notation. For sets A and B we let $A^{\subseteq B} = \bigcup_{C \subseteq B} A^C$. We identify $k \in \mathbb{N}$ with $k = \{0, 1, \dots, k-1\}$. A *partition* of (X, μ) will always mean a finite partition of X into Borel sets. When \mathcal{P} is a partition of (X, μ) we will often identify elements of \mathcal{P} with their equivalence class in MALG_μ . We use the script letters $\mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$ and \mathcal{T} to denote partitions, and the standard italic letters N, O, P, Q, R, S and T respectively to denote their corresponding elements. If \mathcal{P} and \mathcal{Q} are two partitions of (X, μ) then we let $\mathcal{Q} \leq \mathcal{P}$ denote that \mathcal{Q} is a refinement of \mathcal{P} , that is, every $Q \in \mathcal{Q}$ is contained, modulo null sets, in some $P \in \mathcal{P}$. We let $\mathcal{P} \wedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$ denote the greatest common refinement of \mathcal{P} and \mathcal{Q} .

Suppose that $\Gamma \curvearrowright^a (X, \mu)$ and $\mathcal{P} = \{P_0, \dots, P_{k-1}\}$ is a partition of X . If J is a finite subset of Γ and $\tau \in k^J$ then we define

$$P_\tau^a = \bigcap_{\gamma \in J} \gamma^a \cdot P_{\tau(\gamma)}.$$

We will write P_τ when the action a is understood. Note that $P_\emptyset = X$. We let Γ act on the set $\bigcup \{k^J : J \subseteq \Gamma \text{ is finite}\}$ by shift, that is, $(\gamma \cdot \tau)(\delta) = \tau(\gamma^{-1} \delta)$. Then $\text{dom}(\gamma \cdot \tau) = \gamma \text{dom}(\tau)$.

The following lemma is a generalization of [AW13, Lemma 5]. It establishes a sufficient condition for weak containment which will be used in the proof of Theorem 1.5.

LEMMA 5.14. Suppose that $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ and \mathcal{B} is a collection of measure preserving actions of Γ . Suppose that $\mathcal{P}^{(0)} \geq \mathcal{P}^{(1)} \geq \dots$ is a sequence of partitions of X such that the smallest \mathbf{a} -invariant measure algebra containing $\bigcup_n \mathcal{P}^{(n)}$ is all of MALG_μ . Then $\mathbf{a} \prec \mathcal{B}$ if for any n , writing $\mathcal{P}^{(n)} = \{P_0, \dots, P_{k-1}\}$, for all finite subsets $F \subseteq \Gamma$ and all $\delta > 0$, there exist some $\Gamma \curvearrowright^b (Y, \nu) = \mathbf{b} \in \mathcal{B}$ and a partition $\mathcal{Q} = \{Q_0, \dots, Q_{k-1}\}$ of Y such that for all $\tau \in k^{\subseteq F}$, $|\mu(P_\tau) - \nu(Q_\tau)| < \delta$.

Proof. Suppose that the condition is satisfied and let $A_1, \dots, A_m \in \text{MALG}_\mu$, $F_0 \subseteq \Gamma$ finite with $e \in F_0$, and $\epsilon > 0$ be given. Let $e \in G_0 \subseteq G_1 \subseteq \dots$ be an increasing exhaustive sequence of finite subsets of Γ , and let $\mathbf{B}^{(n)}$ denote the Boolean algebra generated by $\bigcup_{\gamma \in G_n} \gamma^a \cdot \mathcal{P}^{(n)}$. Then $\mathbf{B}^{(0)} \subseteq \mathbf{B}^{(1)} \subseteq \dots$, and the Boolean algebra $\bigcup_n \mathbf{B}^{(n)}$ is dense in MALG_μ . We may therefore find some $N \in \mathbb{N}$ and sets $D_1, \dots, D_m \in \mathbf{B}^{(N)}$ such that $\mu(A_i \Delta D_i) < \epsilon/4$ for all $i \leq m$. Let $G = G_N$ and let $\mathcal{P} = \mathcal{P}^{(N)} = \{P_0, \dots, P_{k-1}\}$.

We can express each D_i as a finite disjoint union of sets of the form P_σ , $\sigma \in k^T$, that is, $D_i = \bigsqcup \{P_\sigma : \sigma \in I_i\}$ for some $I_i \subseteq k^G$. Applying the condition given by the lemma to $F = F_0 G$ and $0 < \delta < \epsilon/2k^{|G|}$ we obtain $\Gamma \curvearrowright^b (Y, \nu) = \mathbf{b} \in \mathcal{B}$ and a partition $\mathcal{Q} = \{Q_0, \dots, Q_{k-1}\} \subseteq \text{MALG}_\nu$ such that for all $\tau \in k^{\subseteq F_0 G}$, $|\mu(P_\tau) - \nu(Q_\tau)| < \delta$. For $i \leq m$ we let $B_i = \bigsqcup \{Q_\sigma : \sigma \in I_i\}$. Note that for $\gamma \in F_0$ and $\sigma, \sigma' \in k^G$ we have $\text{dom}(\gamma \cdot \sigma) = \gamma G \subseteq F_0 G$ and

$$\begin{aligned} \gamma^a \cdot P_\sigma \cap P_{\sigma'} &= \begin{cases} P_{\gamma \cdot \sigma \cup \sigma'} & \text{if } \gamma \cdot \sigma \text{ and } \sigma' \text{ are compatible,} \\ \emptyset & \text{otherwise,} \end{cases} \\ \gamma^b \cdot Q_\sigma \cap Q_{\sigma'} &= \begin{cases} Q_{\gamma \cdot \sigma \cup \sigma'} & \text{if } \gamma \cdot \sigma \text{ and } \sigma' \text{ are compatible,} \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

It follows from our choice of F that $|\mu(\gamma^a P_\sigma \cap P_{\sigma'}) - \nu(\gamma^b Q_\sigma \cap Q_{\sigma'})| < \delta$ for all $\sigma, \sigma' \in k^G$. We now have for $i, j \leq m$ and $\gamma \in F_0$ that

$$\begin{aligned} & |\mu(\gamma^a A_i \cap A_j) - \mu(\gamma^b B_i \cap B_j)| \\ & \leq \frac{\epsilon}{2} + \left| \mu \left(\bigsqcup_{\substack{\sigma \in I_i, \\ \sigma' \in I_j}} \gamma^a P_\sigma \cap P_{\sigma'} \right) - \nu \left(\bigsqcup_{\substack{\sigma \in I_i, \\ \sigma' \in I_j}} \gamma^b Q_\sigma \cap Q_{\sigma'} \right) \right| \\ & \leq \frac{\epsilon}{2} + |I_i| |I_j| \delta < \epsilon. \end{aligned} \quad \square$$

5.5. *Independent joinings over an IRS and the proof of Theorem 1.5.* Let $\mathbf{a} = \Gamma \curvearrowright^a (Y, \nu)$ be a non-atomic measure preserving action of Γ , and let $\theta = \text{type}(\mathbf{a})$. The stabilizer map $y \mapsto \Gamma_y$ factors \mathbf{a} onto θ and we let $\nu = \int_H \nu_H d\theta$ be the corresponding disintegration of ν over θ . Fix a standard probability space (Z, η) and let $s_{\theta, \eta} = \Gamma \curvearrowright^s (Z^{\leq \Gamma}, \eta^{\theta \setminus \Gamma})$ be the θ -random Bernoulli shift over (Z, η) . The map $f \mapsto H_f$ factors $s_{\theta, \eta}$ onto θ and the corresponding disintegration is given by $\eta^{\theta \setminus \Gamma} = \int_H \eta^{H \setminus \Gamma} d\theta$. The relatively independent joining of $s_{\theta, \eta}$ and \mathbf{a} over θ is then the action $\Gamma \curvearrowright^{s \times a} (Z^{\leq \Gamma} \times Y, \eta^{\theta \setminus \Gamma} \otimes_\theta \nu)$, where

$$\begin{aligned} \eta^{\theta \setminus \Gamma} \otimes_\theta \nu &= \int_H (\eta^{H \setminus \Gamma} \times \nu_H) d\theta = \int_H \left(\eta^{H \setminus \Gamma} \times \int_{\{y: \Gamma_y = H\}} \delta_y d\nu_H(y) \right) d\theta \\ &= \int_y (\eta^{\Gamma_y \setminus \Gamma} \times \delta_y) d\nu. \end{aligned}$$

It is clear that $\eta^{\theta \setminus \Gamma} \otimes_{\theta} \nu$ concentrates on the set $Z^{\leq \Gamma} \otimes_a Y = \{(f, y) : H_f = \Gamma_y\}$. We write $\mathbf{b} = \Gamma \curvearrowright^b (X, \mu)$ for $\Gamma \curvearrowright^{s \times a} (Z^{\leq \Gamma} \otimes_a Y, \eta^{\theta \setminus \Gamma} \otimes_{\theta} \nu)$, so that $b = s \times a$, $X = Z^{\leq \Gamma} \otimes_a Y$, and

$$\mu = \int_{y \in Y} \eta^{\Gamma_y \setminus \Gamma} \times \delta_y \, d\nu(y).$$

Theorem 1.5 then says that \mathbf{b} is weakly equivalent to \mathbf{a} .

Proof of Theorem 1.5. It suffices to show that $\mathbf{b} < \mathbf{a}$. Let $\mathcal{N}^{(0)} \geq \mathcal{N}^{(1)} \geq \dots$ and $\mathcal{R}^{(0)} \geq \mathcal{R}^{(1)} \geq \dots$ be sequences of finite partitions of Z and Y respectively such that $\bigcup_n \mathcal{N}^{(n)}$ generates MALG_{η} and $\bigcup_n \mathcal{R}^{(n)}$ generates MALG_{ν} (for example, if $Z = Y = 2^{\mathbb{N}}$ then we can let $\mathcal{N}^{(n)} = \mathcal{R}^{(n)}$ consist of the rank n basic clopen sets). For each $\gamma \in \Gamma$ let $\pi_{\gamma} : X \rightarrow Z$ be the projection $\pi_{\gamma}(f, y) = f(\Gamma_y \gamma)$ and define the finite partitions $\mathcal{S}^{(0)} \geq \mathcal{S}^{(1)} \geq \dots$ of X by

$$\mathcal{S}^{(n)} = \{\pi_e^{-1}(N) : N \in \mathcal{N}^{(n)}\}.$$

For $A \subseteq Y$ let $\tilde{A} \subseteq X$ denote the inverse image of A under the projection map $(f, y) \mapsto y \in Y$ and define

$$\tilde{\mathcal{R}}^{(n)} = \{\tilde{R} : R \in \mathcal{R}^{(n)}\}.$$

Then the smallest \mathbf{b} -invariant measure algebra containing the partitions $\mathcal{P}^{(n)} = \mathcal{S}^{(n)} \wedge \tilde{\mathcal{R}}^{(n)}$, $n \in \mathbb{N}$, is all of MALG_{μ} . Fix n , write $\mathcal{N} = \mathcal{N}^{(n)} = \{N_0, \dots, N_{d-1}\}$, and for $i < d$ let

$$\begin{aligned} S_i &= \pi_e^{-1}(N_i), \\ \alpha_i &= \mu(S_i) = \eta(N_i), \end{aligned}$$

along with

$$\begin{aligned} \mathcal{S} &= \mathcal{S}^{(n)} = \{S_0, \dots, S_{d-1}\}, \\ \mathcal{R} &= \mathcal{R}^{(n)} = \{R_0, \dots, R_{k-1}\}, \\ \mathcal{P} &= \mathcal{P}^{(n)} = \{P_{i,j} = S_i \cap \tilde{R}_j : i < d, j < k\}. \end{aligned}$$

For $F \subseteq \Gamma$ finite we naturally identify $(d \times k)^{\subseteq F}$ with $\bigcup_{J \subseteq F} d^J \times k^J$. Under this identification, for $J \subseteq F$ and $(\tau, \sigma) \in d^J \times k^J$ we have

$$\begin{aligned} P_{(\tau, \sigma)}^b &= \bigcap_{\gamma \in J} \gamma^{s \times a} P_{\tau(\gamma), \sigma(\gamma)} = \bigcap_{\gamma \in J} (\gamma^{s \times a} S_{\tau(\gamma)} \cap \gamma^{s \times a} \tilde{R}_{\sigma(\gamma)}) \\ &= \left(\bigcap_{\gamma \in J} \gamma^{s \times a} S_{\tau(\gamma)} \right) \cap \left(\bigcap_{\gamma \in J} \gamma^{s \times a} \tilde{R}_{\sigma(\gamma)} \right) = S_{\tau}^b \cap \tilde{R}_{\sigma}^b. \end{aligned}$$

By Lemma 5.14, to show that $\mathbf{b} < \mathbf{a}$ it suffices to show that for every $F \subseteq \Gamma$ finite, and $\epsilon > 0$, there exists a partition $\mathcal{Q} = \{Q_{i,j} : i < d, j \leq k\}$ of Y such that for all $J \subseteq F$, $(\tau, \sigma) \in d^J \times k^J$,

$$|\mu(S_{\tau} \cap \tilde{R}_{\sigma}) - \nu(Q_{(\tau, \sigma)})| < \epsilon.$$

Fix such an $F \subseteq \Gamma$ finite and $\epsilon > 0$. We will proceed by finding a partition $\mathcal{T} = \{T_0, \dots, T_{d-1}\}$ of Y , and then take $Q_{i,j} = T_i \cap R_j$, in which case we will have

$$Q_{(\tau, \sigma)} = \left(\bigcap_{\gamma \in J} \gamma^a T_{\tau(\gamma)} \right) \cap \left(\bigcap_{\gamma \in J} \gamma^a R_{\sigma(\gamma)} \right) = T_{\tau} \cap R_{\sigma}.$$

We are therefore looking for a partition \mathcal{T} so that

$$\forall(\tau, \sigma) \in (d \times k)^{\subseteq F}, \quad |\mu(S_\tau \cap \tilde{R}_\sigma) - \nu(T_\tau \cap R_\sigma)| < \epsilon. \quad (5.1)$$

We first calculate the value of $\mu(S_\tau \cap \tilde{A})$ for $\tau \in d^J$ ($J \subseteq F$) and $A \subseteq Y$. Let \mathcal{E}_J denote the finite collection of all equivalence relations on the set J . For $E \in \mathcal{E}_J$ let us say that $\tau \in d^J$ *respects* E , written $\tau \ll E$, if τ is constant on each E -equivalence class. For a subgroup $H \leq \Gamma$ let $E_J(H) \in \mathcal{E}_J$ denote the equivalence relation determined by $tE_J(H)s$ if and only if $Ht = Hs$ (if and only if $t^{-1}H = s^{-1}H$). We write $E_J(y)$ for $E_J(\Gamma_y)$. For each $E \in \mathcal{E}_J$ we fix a transversal $T_E \subseteq J$ for E . We then have

$$\begin{aligned} \mu(S_\tau \cap \tilde{A}) &= \int_{y \in A} \eta^{\Gamma_y \setminus \Gamma}(\{f \in Z^{\Gamma_y \setminus \Gamma} : \forall t \in J (f(\Gamma_y t) \in N_{\tau(t)})\}) d\nu(y) \\ &= \sum_{\{E \in \mathcal{E}_J : \tau \ll E\}} \int_{\{y \in A : E_J(y) = E\}} \eta^{\Gamma_y \setminus \Gamma}(\{f \in Z^{\Gamma_y \setminus \Gamma} : \forall t \in T_E (f(\Gamma_y t) \in N_{\tau(t)})\}) d\nu(y) \\ &= \sum_{\{E \in \mathcal{E}_J : \tau \ll E\}} \nu(A \cap \{y : E_J(y) = E\}) \prod_{t \in T_E} \alpha_{\tau(t)}. \end{aligned} \quad (5.2)$$

We now proceed as in the proof of [AW13, Theorem 1]. Without loss of generality Y is a compact metric space with compatible metric $d_Y \leq 1$. Fix some $\epsilon_0 > 0$ such that $\epsilon_0^{1/2} < \epsilon/(2(dk)^{|F|/2}2^{|F|+1})$. For $\delta \geq 0$ define the sets

$$\begin{aligned} D_\delta &= \{y \in Y : \forall s, t \in F (t^{-1}y \neq s^{-1}y \Rightarrow d_Y(t^{-1}y, s^{-1}y) > \delta)\}, \\ E_\delta &= \{(y, y') \in D_\delta \times D_\delta : \forall s, t \in F (d_Y(s^{-1}y, t^{-1}y') > \delta)\}. \end{aligned}$$

Then $\nu(D_0) = 1$ by definition, and $\nu^2(E_0) = 1$ since ν is non-atomic. Thus there exists $\delta > 0$ such that

$$\nu(D_\delta) > 1 - \frac{\epsilon_0}{4|\mathcal{E}_F|} \quad \text{and} \quad \nu^2(E_\delta) > 1 - \frac{\epsilon_0}{4|\mathcal{E}_F|^2}.$$

Fix a finite Borel partition $\{O_m : 1 \leq m \leq M\}$ of Y with $\text{diam}(O_m) < \delta$ for each m . For $y \in Y$ let $\alpha(y) = m$ if and only if $y \in O_m$. Let $(\Omega, \mathbb{P}) = (d^M, \rho^M)$ and let $Y_m(\omega) = \omega(m)$, so that $\{Y_m : 1 \leq m \leq M\}$ are independent and identically distributed random variables. For $\omega \in \Omega$ and $i = 0, \dots, d-1$ define

$$T_i(\omega) = \{y \in Y : \omega(\alpha(y)) = i\}.$$

Then each $\omega \in \Omega$ defines the partition $\mathcal{T}(\omega) = \{T_0(\omega), \dots, T_{d-1}(\omega)\}$ of Y . Let $T_i = \{(\omega, y) : y \in T_i(\omega)\}$ and let

$$T_\tau = \left\{ (\omega, y) \in \Omega \times Y : y \in T_\tau(\omega) = \bigcap_{t \in J} t^a \cdot (T_{\tau(t)}(\omega)) \right\}, \quad \tau \in d^{\subseteq F}.$$

We view \mathcal{T} as a ‘random partition’ of Y . We let Γ act on Ω trivially so that, for example, $\gamma \cdot (T_\tau(\omega)) = (\gamma \cdot T_\tau)(\omega)$, and for $B \subseteq \Omega \times Y$ and $y \in Y$ we let B^y denote the section $B^y = \{\omega : (\omega, y) \in B\}$. We show that \mathcal{T} satisfies (5.1) with high probability.

Now fix some $A \subseteq Y$ and $\tau \in d^J$, $J \subseteq F$. Note that if $y \in Y$ and τ does not respect $E_J(y)$ then there exist $t, s \in J$ with $t^{-1}y = s^{-1}y$ and $\tau(t) \neq \tau(s)$, so that $(T_{\tau(t)})^{t^{-1}y} \cap (T_{\tau(s)})^{s^{-1}y} = \emptyset$ and thus $(T_\tau)^y = \bigcap_{t \in J} (t \cdot T_{\tau(t)})^y = \bigcap_{t \in J} (T_{\tau(t)})^{t^{-1}y} = \emptyset$. It follows that

the expected measure of $T_\tau(\omega) \cap A$ is

$$\begin{aligned}
 \mathbb{E}[v(T_\tau(\omega) \cap A)] &= \int_A \left(\int_\Omega 1_{T_\tau}(\omega, y) d\mathbb{P}(\omega) \right) dv(y) \\
 &= \int_A \mathbb{P}((T_\tau)^y) dv(y) = \sum_{\{E \in \mathcal{E}_J : \tau \ll E\}} \int_{\{y \in A : E_J(y) = E\}} \mathbb{P}((T_\tau)^y) dv(y) \\
 &= \sum_{\{E \in \mathcal{E}_J : \tau \ll E\}} \left(\int_{\{y \in A \cap D_\delta : E_J(y) = E\}} \mathbb{P}((T_\tau)^y) dv \right) + \int_{A \setminus D_\delta} \mathbb{P}((T_\tau)^y) dv. \quad (5.3)
 \end{aligned}$$

Fix some $E \in \mathcal{E}_J$ with $\tau \ll E$ and some $y \in D_\delta$ with $E_J(y) = E$. For $t, s \in J$, if t and s are not E -related then $t^{-1}y \neq s^{-1}y$ and so $d_Y(t^{-1}y, s^{-1}y) > \delta$. It follows that $O_{\alpha(t^{-1}y)} \neq O_{\alpha(s^{-1}y)}$ since each O_α has diameter smaller than δ . So as t ranges over T_E , the numbers $\alpha(t^{-1}y)$ are all distinct and the variables $Y_{\alpha(t^{-1}y)} : \omega \mapsto \omega(\alpha(t^{-1}y))$, $t \in T_E$, are therefore independent. We have $t^{-1}y \in T_{\tau(t)}(\omega)$ if and only if $\omega(\alpha(t^{-1}y)) = \tau(t)$, so the sets $(t \cdot T_{\tau(t)})^y = (T_{\tau(t)})^{t^{-1}y}$, $t \in T_E$, are all independent. If tEs then as $\tau \ll E$ we have that $(T_{\tau(t)})^{t^{-1}y} = (T_{\tau(s)})^{s^{-1}y}$. It follows that

$$\mathbb{P}((T_\tau)^y) = \mathbb{P}\left(\bigcap_{t \in J} (t \cdot T_{\tau(t)})^y\right) = \prod_{t \in T_E} \mathbb{P}((T_{\tau(t)})^{t^{-1}y}) = \prod_{t \in T_E} \alpha_{\tau(t)}. \quad (5.4)$$

Continuing the computation, the second integral in (5.3) is no greater than $v(A \setminus D_\delta) < \epsilon_0/4$ and $v(A \cap D_\delta \cap \{y : E_J(y) = E\})$ is within $\epsilon_0/4|\mathcal{E}_F|$ of $v(A \cap \{y : E_J(y) = E\})$, so after summing over all $E \in \mathcal{E}_J$ we see that (5.3) is within $\epsilon_0/2$ of (5.2), that is,

$$|\mathbb{E}[v(T_\tau(\omega) \cap A)] - \mu(S_\tau \cap \tilde{A})| < \frac{\epsilon_0}{2}. \quad (5.5)$$

Now we compute the second moment of $v(T_\tau(\omega) \cap A)$:

$$\begin{aligned}
 \mathbb{E}[v(T_\tau(\omega) \cap A)^2] &= \int_\Omega \left(\int_{y \in A} 1_{T_\tau}(\omega, y) dv(y) \right) \left(\int_{y' \in A} 1_{B_\tau}(\omega, y') dv(y') \right) d\mathbb{P} \\
 &= \int_{(y, y') \in A \times A} \left(\int_\Omega 1_{T_\tau}(\omega, y) 1_{T_\tau}(\omega, y') d\mathbb{P} \right) dv^2 \\
 &= \int_{(y, y') \in A \times A} \mathbb{P}((T_\tau)^y \cap (T_\tau)^{y'}) dv^2. \quad (5.6)
 \end{aligned}$$

For $(y, y') \in E_\delta$, if $t, s \in J$ then $d_Y(t^{-1}y, s^{-1}y') > \delta$, so that $O_{\alpha(t^{-1}y)}$ and $O_{\alpha(s^{-1}y')}$ are disjoint. It follows that the two events

$$\{\omega : \forall t \in J (Y_{\alpha(t^{-1}y)}(\omega) = \tau(t))\} = \bigcap_{t \in J} (T_{\tau(t)})^{t^{-1}y} = (T_\tau)^y$$

and

$$\{\omega : \forall s \in J (Y_{\alpha(s^{-1}y')}(\omega) = \tau(s))\} = \bigcap_{s \in J} (T_{\tau(s)})^{s^{-1}y'} = (T_\tau)^{y'}$$

are independent. We obtain that the part of (5.6) integrated over $(A \times A) \cap E_\delta$ is equal to

$$\begin{aligned} & \int_{(y, y') \in (A \times A) \cap E_\delta} \mathbb{P}((T_\tau)^y \cap (T_\tau)^{y'}) dv^2 \\ &= \int_{(y, y') \in (A \times A) \cap E_\delta} \mathbb{P}((T_\tau)^y) \mathbb{P}((T_\tau)^{y'}) dv^2 \\ &= \sum_{\tau \ll E, E' \in \mathcal{E}_J} v^2((A \times A) \cap E_\delta \cap \{(y, y') : E_J(y) = E, E_J(y') = E'\}) \prod_{t \in T_E} \alpha_{\tau(t)} \\ & \quad \times \prod_{s \in T_{E'}} \alpha_{\tau(s)}, \end{aligned}$$

where we used the fact that $E_\delta \subseteq D_\delta \times D_\delta$ along with the known values from (5.3) and (5.4). The part of (5.6) integrated over $(A \times A) \setminus E_\delta$ is no greater than $\epsilon_0/4$, and for each pair $E, E' \in \mathcal{E}_J$ with $\tau \ll E, E'$, the value of $v^2((A \times A) \cap E_\delta \cap \{(y, y') : E_J(y) = E, E_J(y') = E'\})$ is within $\epsilon_0/4|\mathcal{E}_F|^2$ of $v(A \cap \{y : E_J(y) = E\})v(A \cap \{y' : E_J(y') = E'\})$. Summing over all such $E, E' \in \mathcal{E}_J$ we obtain that (5.6) is within $\epsilon_0/2$ of the square of (5.2), that is,

$$|\mathbb{E}[v(T_\tau(\omega) \cap A)^2] - \mu(S_\tau \cap \tilde{A})^2| < \frac{\epsilon_0}{2}. \quad (5.7)$$

From (5.5) and (5.7) it follows that the variance of $v(T_\tau(\omega) \cap A)$ is no greater than ϵ_0 . By Chebyshev's inequality we then have

$$\begin{aligned} \mathbb{P}(|v(T_\tau(\omega) \cap A) - \mu(S_\tau \cap \tilde{A})| \geq \epsilon) &\leq \mathbb{P}\left(|v(T_\tau(\omega) \cap A) - \mathbb{E}[v(T_\tau(\omega) \cap A)]| \geq \frac{\epsilon}{2}\right) \\ &\leq \mathbb{P}(|v(T_\tau(\omega) \cap A) - \mathbb{E}[v(T_\tau(\omega) \cap A)]| \geq (kd)^{|F|/2} 2^{|F|+1} \epsilon_0^{1/2}) \leq \frac{1}{(kd)^{|F|} 2^{2|F|+2}}, \end{aligned}$$

and since this is true for each $\tau \in d^{\subseteq F}$ and $|d^{\subseteq F}| \leq 2^{|F|} d^{|F|}$, we find that

$$\mathbb{P}(\exists \tau \in d^{\subseteq F} (|v(T_\tau(\omega) \cap A) - \mu(S_\tau \cap \tilde{A})| \geq \epsilon)) \leq \frac{1}{2^{|F|+2} k^{|F|}}.$$

Since $A \subseteq Y$ was arbitrary, this is in particular true for each $A = R_\sigma, \sigma \in k^{\subseteq F}$, so that

$$\mathbb{P}(\exists \tau \in d^{\subseteq F}, \sigma \in k^{\subseteq F} (|v(T_\tau(\omega) \cap R_\sigma) - \mu(S_\tau \cap \tilde{R}_\sigma)| > \epsilon)) \leq \frac{1}{4}.$$

So taking any ω_0 in the complement of the above set, we obtain a partition $\mathcal{T} = \mathcal{T}(\omega_0)$ satisfying (5.1). \square

Theorem 1.5 shows that among all non-atomic weak equivalence classes of type θ there is a least, in the sense of weak containment, namely $s_{\theta, \lambda}$ where λ is Lebesgue measure on $[0, 1]$. We note that there is also a greatest.

THEOREM 5.15. *Let $\theta \in \text{IRS}(\Gamma)$. Then there exists a measure preserving action \mathbf{a}_θ of Γ with $\text{type}(\mathbf{a}_\theta) = \theta$ such that for all measure preserving actions \mathbf{b} of Γ , if $\text{type}(\mathbf{b}) = \theta$ then $\mathbf{b} \prec \mathbf{a}_\theta$.*

Proof. Let (Y, ν) be a non-atomic standard probability space. If \mathbf{b} is any measure preserving action of Γ of type θ then $\iota \times \mathbf{b}$ is also of type θ , weakly contains \mathbf{b} , and is

isomorphic to an element of $A(\Gamma, Y, \nu)$. It thus suffices to show there is an action \mathbf{a}_θ of type θ that weakly contains every element in the set $\mathcal{A}_\theta = \{\mathbf{a} \in A(\Gamma, Y, \nu) : \text{type}(\mathbf{a}) = \theta\}$.

Let $\{\mathbf{a}_n\}_{n \in \mathbb{N}}$ be a countable dense subset of \mathcal{A}_θ . For each n the stabilizer map $y \mapsto \text{stab}_{\mathbf{a}_n}(y) = \{\gamma \in \Gamma : \gamma^{a_n} y = y\}$ factors \mathbf{a}_n onto θ . Let \mathbf{a}_θ denote the relatively independent joining of the actions $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \dots$ over the common factor θ , that is, $\mathbf{a}_\theta = \Gamma \curvearrowright \prod_n \mathbf{a}_n$ ($Y^\mathbb{N}, \nu_\theta$), where the measure ν_θ has each marginal equal to ν and concentrates on the set $\{(y_0, y_1, y_2, \dots) \in Y^\mathbb{N} : \forall n (\text{stab}_{\mathbf{a}_n}(y_n) = \text{stab}_{\mathbf{a}_0}(y_0))\}$. Then for ν_θ -almost every $(y_0, y_1, \dots) \in Y^\mathbb{N}$ we have $\text{stab}_{\prod_n \mathbf{a}_n}((y_0, y_1, \dots)) = \text{stab}_{\mathbf{a}_0}(y_0)$, from which it follows that $\text{type}(\mathbf{a}_\theta) = \theta$. Since $\mathbf{a}_n \sqsubseteq \mathbf{a}_\theta$ for all n the set $\{\mathbf{a} \in \mathcal{A}_\theta : \mathbf{a} < \mathbf{a}_\theta\}$ is dense in \mathcal{A}_θ so, by Proposition 3.5, \mathbf{a}_θ weakly contains every element of \mathcal{A}_θ . \square

6. Non-classifiability

6.1. *Non-classifiability by countable structures of \cong , \cong^w , and \cong^u on free weak equivalence classes.*

Definition 6.1. Let E and F be equivalence relations on the standard Borel spaces X and Y respectively.

- (1) A *homomorphism* from E to F is a map $\psi : X \rightarrow Y$ such that $x E y \Rightarrow \psi(x) F \psi(y)$.
- (2) A *reduction* from E to F is a map $\psi : X \rightarrow Y$ such that $x E y \Leftrightarrow \psi(x) F \psi(y)$.
- (3) E is said to *admit classification by countable structures* if there exist a countable language \mathcal{L} and a Borel reduction from E to isomorphism $\cong_{\mathcal{L}}$ on $X_{\mathcal{L}}$, where $X_{\mathcal{L}}$ is the space of all \mathcal{L} -structures with universe \mathbb{N} .
- (4) Suppose that the space X is Polish. We say that E is *generically F -ergodic* if for every Baire measurable homomorphism ψ from E to F , there exists some $y \in Y$ such that $\psi^{-1}([y]_F)$ is comeager.

The proof of the following lemma is clear.

LEMMA 6.2. *Let F_1 and F_2 be equivalence relations on the standard Borel spaces Y_1 and Y_2 respectively, and let E be an equivalence relation on the Polish space P . Suppose that E is generically F_2 -ergodic and that there exists a Borel reduction from F_1 to F_2 . Then E is generically F_1 -ergodic.*

Since the orbit equivalence relation associated to a generically turbulent Polish group action is generically $\cong_{\mathcal{L}}$ -ergodic for all countable languages \mathcal{L} [Hj00], Lemma 6.2 immediately implies the following result.

LEMMA 6.3. *Let \mathcal{G} be a Polish group and let P be a generically turbulent Polish \mathcal{G} -space with corresponding orbit equivalence relation $E_{\mathcal{G}}^P$. Let F be an equivalence relation on a standard Borel space Y and suppose that $E_{\mathcal{G}}^P$ is not generically F -ergodic. Then F does not admit classification by countable structures.*

Let \mathcal{H} be an infinite dimensional separable Hilbert space and let $\mathcal{U}(\mathcal{H})$ denote the unitary group of \mathcal{H} which is a Polish group under the strong operator topology. The group $\mathcal{U}(\mathcal{H})$ acts on $\mathcal{U}(\mathcal{H})^\Gamma$ by conjugation on each coordinate and we may view the space $\text{Rep}(\Gamma, \mathcal{H})$ of all unitary representations of Γ on \mathcal{H} as an invariant closed subspace of $\mathcal{U}(\mathcal{H})^\Gamma$, so that it is a Polish $\mathcal{U}(\mathcal{H})$ -space. We call the corresponding orbit equivalence

relation on $\text{Rep}(\Gamma, \mathcal{H})$ unitary conjugacy, and if π_1 and π_2 are in the same unitary conjugacy class then we say that π_1 and π_2 are unitarily conjugate and write $\pi_1 \cong \pi_2$. Let $\lambda_\Gamma : \Gamma \rightarrow \mathcal{U}(\ell_2(\Gamma))$ denote the left regular representation of Γ and let $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ be the set of unitary representations of Γ on $\mathcal{U}(\mathcal{H})$ that are weakly contained in λ_Γ . Then $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ is also a Polish $\mathcal{U}(\mathcal{H})$ space, being an invariant closed subspace of $\text{Rep}(\Gamma, \mathcal{H})$.

The following lemma is proved in the same way as [KLP10, Lemma 2.4], using that the reduced dual $\hat{\Gamma}_\lambda$, which may be identified with the spectrum of the reduced C^* -algebra $C_\lambda^*(\Gamma)$, contains no isolated points [KLP10, 3.2].

LEMMA 6.4. *Let κ be a unitary representation of Γ and let \mathcal{H} be an infinite dimensional separable Hilbert space. Then the set $\{\pi \in \text{Rep}_\lambda(\Gamma, \mathcal{H}) : \pi \perp \kappa\}$ is dense G_δ in $\text{Rep}_\lambda(\Gamma, \mathcal{H})$.*

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Given a free action $\mathbf{a}_0 \in A(\Gamma, X, \mu)$, we let $[\mathbf{a}_0] = \{\mathbf{b} \in A(\Gamma, X, \mu) : \mathbf{b} \sim \mathbf{a}_0\}$ denote its weak equivalence class in $A(\Gamma, X, \mu)$. Let $\mathcal{H} = \ell^2(\Gamma)$ and let $\mathbf{g} : \text{Rep}(\Gamma, \mathcal{H}) \rightarrow A(\Gamma, X, \mu)$ be the continuous map assigning to each $\pi \in \text{Rep}_\lambda(\Gamma, \mathcal{H})$ the corresponding Gaussian action $\mathbf{g}(\pi) \in A(\Gamma, X, \mu)$ (see [Ke10, Appendix E]). We have that $\mathbf{g}(\pi) \prec \mathbf{g}(\infty \cdot \lambda_\Gamma) \cong s_\Gamma$ and so, by Corollary 1.6, $\mathbf{a}_0 \times \mathbf{g}(\pi) \sim \mathbf{a}_0$. Fix some isomorphism $\varphi : X^2 \rightarrow X$ of the measure spaces (X^2, μ^2) and (X, μ) and denote by $\mathbf{b} \mapsto \varphi \cdot \mathbf{b}$ the corresponding homeomorphism of $A(\Gamma, X^2, \mu^2)$ with $A(\Gamma, X, \mu)$. Let $\psi : \text{Rep}_\lambda(\Gamma, \mathcal{H}) \rightarrow [\mathbf{a}_0]$ be the map $\pi \mapsto \varphi \cdot (\mathbf{a}_0 \times \mathbf{g}(\pi))$. This is a continuous homomorphism from unitary conjugacy on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ to isomorphism on $[\mathbf{a}_0]$, and is therefore also a homomorphism to \cong^w and to \cong^u on $[\mathbf{a}_0]$.

CLAIM. *The inverse image under ψ of each unitary equivalence class in $[\mathbf{a}_0]$ is meager. In particular, the same is true for each isomorphism class and each weak isomorphism class.*

Proof of claim. Let $\mathbf{c} \in [\mathbf{a}_0]$. By Lemma 6.4 the set $\{\pi \in \text{Rep}_\lambda(\Gamma, \mathcal{H}) : \pi \perp \kappa_0^c\}$ is comeager in $\text{Rep}_\lambda(\Gamma, \mathcal{H})$. If $\psi(\pi) \cong^u \mathbf{c}$ then $\pi \leq \kappa_0^{g(\pi)} \leq \kappa_0^{a_0 \times g(\pi)} \cong \kappa_0^c$, so that $\pi \not\perp \kappa_0^c$. \square

By [KLP10, 3.3], the conjugacy action of $\mathcal{U}(\mathcal{H})$ on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ is generically turbulent. The homomorphism ψ witnesses that unitary conjugacy on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ is not generically $F|[\mathbf{a}_0]$ -ergodic when F is any of \cong , \cong^w , or \cong^u . The theorem now follows from Lemma 6.3. \square

Remark 6.5. If the weak equivalence class $[\mathbf{a}_0]$ contains an ergodic (respectively, weak mixing) action \mathbf{b}_0 , then the action $\mathbf{b}_0 \times \mathbf{g}(\pi)$ is ergodic (respectively, weak mixing) provided that the representation $\pi \in \text{Rep}_\lambda(\Gamma, \mathcal{H})$ is weak mixing. Since the weak mixing π are dense G_δ in $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ [KLP10, 3.6] we conclude that isomorphism (and \cong^w and \cong^u) restricted to the ergodic (respectively, weak mixing) elements of $[\mathbf{a}_0]$ does not admit classification by countable structures.

It also follows from the above arguments and [HK95, 2.2] that the equivalence relation E_0 of eventual agreement on $2^\mathbb{N}$ is Borel reducible to $F|[\mathbf{a}_0]$ when F is any of \cong , \cong^w , or \cong^u (and the same holds for $F|\{\mathbf{b} \in [\mathbf{a}_0] : \mathbf{b} \text{ is ergodic (respectively, weak mixing)}\}$ when $[\mathbf{a}_0]$ contains ergodic (respectively, weak mixing) elements).

6.2. Extending Theorem 1.7. It would be interesting to see an extension of Theorem 1.7 to weak equivalence classes of measure preserving actions that are not necessarily free. We outline here one possible generalization of the argument given in the proof of Theorem 1.7 to measure preserving actions that almost surely have infinite orbits. Let $\alpha = \Gamma \curvearrowright^a (X, \mu)$ be such an action, and let $\theta = \text{type}(\alpha)$, so that θ concentrates on the infinite index subgroups of Γ . In place of unitary conjugacy on $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ we work with the cohomology equivalence relation on a certain orbit closure in the Polish space $Z^1(\theta, \mathcal{U}(\mathcal{H}))$ of unitary cocycles of θ , where $\mathcal{H} = \ell^2(\mathbb{N})$. The cohomology equivalence relation on $Z^1(\theta, \mathcal{U}(\mathcal{H}))$ is the orbit equivalence relation generated by the action of the Polish group $\widetilde{\mathcal{U}(\mathcal{H})} = L(\text{Sub}(\Gamma), \theta, \mathcal{U}(\mathcal{H}))$ given by

$$(f \cdot \alpha)(\gamma, H) = f(\gamma H \gamma^{-1}) \alpha(\gamma, H) f(H)^{-1} \in \mathcal{U}(\mathcal{H}),$$

where $f \in \widetilde{\mathcal{U}(\mathcal{H})}$, $\alpha \in Z^1(\theta, \mathcal{U}(\mathcal{H}))$, $\gamma \in \Gamma$, and $H \leq \Gamma$ (see [Ke10, Ch. III]). In place of the left regular representation λ of Γ we use a cocycle λ_θ associated to θ defined as follows. Identify right cosets of the infinite index subgroups $H \leq \Gamma$ with natural numbers by fixing a Borel map $n : \text{Sub}(\Gamma) \times \Gamma \rightarrow \mathbb{N}$ such that for each infinite index $H \leq \Gamma$ the map $\gamma \mapsto n(H, \gamma)$ is a surjection onto \mathbb{N} and satisfies $n(H, \gamma) = n(H, \delta)$ if and only if $H\gamma = H\delta$. Let $\{e_n\}_{n \in \mathbb{N}}$ be the standard orthonormal basis for $\ell^2(\mathbb{N}) = \mathcal{H}$ and define $\lambda_\theta \in Z^1(\theta, \mathcal{U}(\mathcal{H}))$ by

$$\lambda_\theta(\gamma, H)(e_{n(H, \delta)}) = e_{n(\gamma H \gamma^{-1}, \gamma \delta)}$$

for all $\gamma \in \Gamma$ and $H \leq \Gamma$ of infinite index (recall that θ -almost every H is infinite index in Γ). Fix an isomorphism $T : \infty \cdot \mathcal{H} \rightarrow \mathcal{H}$ and let $\sigma \in Z^1(\theta, \mathcal{U}(\mathcal{H}))$ be the image of $\infty \cdot \lambda_\theta$ under T , that is, $\sigma(\gamma, H) = T \circ (\infty \cdot \lambda_\theta)(\gamma, H) \circ T^{-1}$. Let $Z_\lambda^1(\theta, \mathcal{U}(\mathcal{H}))$ denote the orbit closure of σ in $Z^1(\theta, \mathcal{U}(\mathcal{H}))$. Using the Gaussian map $U(\mathcal{H}) \rightarrow \text{Aut}(X, \mu)$ (see [Ke10, Appendix E] or [BT-D13]), each $\alpha \in Z_\lambda^1(\theta, \mathcal{U}(\mathcal{H}))$ gives rise to a cocycle $g(\alpha) : \Gamma \times \text{Sub}(\Gamma) \rightarrow \text{Aut}(X, \mu)$ of θ with values in the automorphism group $\text{Aut}(X, \mu)$ of a non-atomic probability space (X, μ) . We obtain a skew product action $\mathbf{g}(\alpha) = (X, \mu) \rtimes_{g(\alpha)} \theta$ on the measure space $(Y, \nu) = (X \times \text{Sub}(\Gamma), \mu \times \theta)$, which is an extension of θ . The action $\mathbf{g}(\lambda_\theta)$ is isomorphic to $s_{\theta, \eta}$ (where η is non-atomic) and so the action $\mathbf{g}(\sigma)$ is isomorphic to $s_{\theta, \eta} \cong s_{\theta, \eta}$ as well. Since $\alpha \in Z_\lambda^1(\theta, \mathcal{U}(\mathcal{H}))$ we have $\mathbf{g}(\alpha) \prec s_{\theta, \eta}$ and thus the relatively independent joining $\mathbf{g}(\alpha) \otimes_\theta \mathbf{a}$ is weakly equivalent to \mathbf{a} by Theorem 1.5. The map $\psi_\theta(\alpha) := \varphi \cdot (\mathbf{g}(\alpha) \otimes_\theta \mathbf{a})$ is then a homomorphism from the cohomology equivalence relation on $Z_\lambda^1(\theta, \mathcal{U}(\mathcal{H}))$ to isomorphism on $[\mathbf{a}]$, where $\varphi : Y \times X \rightarrow X$ is once again an isomorphism of measure spaces. The remaining ingredient that is needed is an analogue of the results from [KLP10].

Question 6.6. Let θ be an ergodic IRS of Γ with infinite index. Is the action of $\widetilde{\mathcal{U}(\mathcal{H})}$ on the space $Z_\lambda^1(\theta, \mathcal{U}(\mathcal{H}))$ generically turbulent? Is the preimage under ψ_θ of each \cong^u -class meager?

Two ergodic theoretic analogues of the space $\text{Rep}_\lambda(\Gamma, \mathcal{H})$ are the spaces $A_0(\Gamma, X, \mu) = \{\mathbf{a} \in A(\Gamma, X, \mu) : \mathbf{a} \prec s_\Gamma\}$ and $A_1(\Gamma, X, \mu) = \{\mathbf{a} \in A(\Gamma, X, \mu) : \mathbf{a} \prec_s s_\Gamma\}$, where (X, μ) is non-atomic. When Γ is amenable it follows from [FW04] that these spaces both coincide with $A(\Gamma, X, \mu)$ and the conjugacy action of $\text{Aut}(X, \mu)$ on $A(\Gamma, X, \mu)$ is generically

turbulent. For non-amenable Γ , the spaces $A_0(\Gamma, X, \mu)$, $A_1(\Gamma, X, \mu)$ and $A(\Gamma, X, \mu)$ do not all coincide.

Question 6.7. Let Γ be a non-amenable group. Is conjugacy on either of $A_0(\Gamma, X, \mu)$ or $A_1(\Gamma, X, \mu)$ generically turbulent?

For all non-amenable Γ the set $A_0(\Gamma, X, \mu)$ is nowhere dense in $A_1(\Gamma, X, \mu)$ (by Theorem 1.3), so these two spaces may behave quite differently, generically (indeed, every action in $A_0(\Gamma, X, \mu)$ is ergodic, while the generic action in $A_1(\Gamma, X, \mu)$ has continuous ergodic decomposition). The question of generic turbulence of conjugacy on $\text{ERG}(\Gamma, X, \mu) = \{a \in A(\Gamma, X, \mu) : a \text{ is ergodic}\}$ is discussed in [Ke10, §§5 and 12].

7. Types and amenability

As noted in Remark 4.1, any two free measure preserving actions of an infinite amenable group Γ are weakly equivalent. In this section we prove Theorem 1.8, which extends this to actions that are not necessarily free.

7.1. The space $\text{COS}(\Gamma)$. Let $\text{COS}(\Gamma)$ be the space of all left cosets of all subgroups of Γ . Since $F \in \text{COS}(\Gamma) \Leftrightarrow \forall \delta \in \Gamma (\delta \in F \Rightarrow \delta^{-1}F \in \text{Sub}(\Gamma))$ it follows that $\text{COS}(\Gamma)$ is a closed subset of 2^Γ . As every left coset of a subgroup $H \leq \Gamma$ is equal to a right coset of a conjugate of H and vice versa, $\text{COS}(\Gamma)$ is also the space of all right cosets of subgroups of Γ and we have the equality $\text{COS}(\Gamma) = \{\gamma H \delta^{-1} : H \leq \Gamma, \gamma, \delta \in \Gamma\} \subseteq 2^\Gamma$. We let ℓ denote the continuous action of Γ on $\text{COS}(\Gamma)$ by left translation, $\gamma^\ell \cdot (H\delta) = \gamma H \delta$.

LEMMA 7.1. *Let Γ be a countable amenable group and let $a = \Gamma \curvearrowright^a (X, \mu)$ be a measure preserving action of Γ . Then for any finite $F \subseteq \Gamma$ and $\delta > 0$ there exists a measurable map $J : X \rightarrow \text{COS}(\Gamma)$ such that*

$$\mu(\{x \in X : \forall \gamma \in F J(\gamma^a x) = \gamma^\ell \cdot J(x)\}) \geq 1 - \delta$$

and $J(x) \in \Gamma_x \backslash \Gamma$ for all x .

Proof. We note that this is a generalized version of [BT-D13, Theorem 3.1] which applies to the case in which a is free and which is an immediate consequence of the Rokhlin lemma for free actions of amenable groups. For the general case we use the Ornstein–Weiss theorem [OW80, Theorem 6] which implies that the orbit equivalence relation E_a generated by a is hyperfinite when restricted to an invariant conull Borel set $X' \subseteq X$. We may assume without loss of generality that $X' = X$ and E_a is hyperfinite. Then there exists an increasing sequence $E_0 \subseteq E_1 \subseteq \dots$ of finite Borel sub-equivalence relations of E_a such that $E_a = \bigcup_{n=0}^\infty E_n$. Let F and $\delta > 0$ be given and find $N \in \mathbb{N}$ large enough so that $\mu(X_N) > 1 - \delta$ where $X_N = \{x : \gamma^a x \in [x]_{E_N} \text{ for all } \gamma \in F\}$. Fix a Borel selector $s : X \rightarrow X$ for E_N , that is, for all $x, y \in [x]_{E_N}$ we have $s(x) = s(y)$, and let $x \mapsto \gamma_x \in \Gamma$ be any Borel map such that $\gamma_x^a \cdot s(x) = x$ for all $x \in X$. Define $J : X \rightarrow \text{COS}(\Gamma)$ by $J(x) = \gamma_x \Gamma_{s(x)}$. Then $J(x) \in \Gamma_x \backslash \Gamma$ since $\Gamma_x = \Gamma_{\gamma_x^a \cdot s(x)} = \gamma_x \Gamma_{s(x)} \gamma_x^{-1}$. For each $x \in X_N$ and $\gamma \in F$ we have $\gamma^a x \in [x]_{E_N}$ so that $s(\gamma^a x) = s(x)$ and thus $(\gamma_{\gamma^a x})^a \cdot s(x) = \gamma^a x = (\gamma \gamma_x)^a \cdot s(x)$. It follows that

$$J(\gamma^a x) = \gamma_{\gamma^a x} \Gamma_{s(x)} = \gamma \gamma_x \Gamma_{s(x)} = \gamma^\ell \cdot J(x). \quad \square$$

7.2. Proof of Theorem 1.8.

Proof of Theorem 1.8(1). Since $\text{type}(\mathbf{a})$ is an invariant of stable weak equivalence (see Remark 5.8), it remains to show the following statement.

(*) If $\theta \in \text{IRS}(\Gamma)$ and \mathbf{a} and \mathbf{d} are measure preserving actions of Γ , both of type θ , then

$$\mathbf{a} \sim_s \mathbf{d}.$$

We first show that (*) holds under the assumption that \mathbf{a} and \mathbf{d} are both ergodic. For this, by Theorem 1.5 it suffices to show that for any ergodic measure preserving action $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ of Γ , if $\text{type}(\mathbf{a}) = \theta$ then $\mathbf{a} < s_{\theta, \eta}$ for some standard probability space (Z, η) .

We will define a measure preserving action \mathbf{b} containing θ as a factor, and show that the relatively independent joining $\mathbf{b} \otimes_{\theta} s_{\theta, \eta}$ weakly contains \mathbf{a} when η is a standard non-atomic probability measure. Then we will be done once we show that $\mathbf{b} \otimes_{\theta} s_{\theta, \eta} \cong s_{\theta, \eta}$.

Let $\mu = \int_H \mu_H d\theta$ be the disintegration of μ via $x \mapsto \text{stab}_a(x)$, and define the measure ν on the space $Y = \bigsqcup_{H \in \text{Sub}(\Gamma)} \{f \in X^{H \setminus \Gamma} : \text{stab}_a(f(H\delta)) = H \text{ for all } \delta \in \Gamma\} \subseteq X^{\leq \Gamma}$ by the equation $\nu = \int_H \mu_H^{H \setminus \Gamma} d\theta$. Let $a^{\leq \Gamma}$ be the action on $X^{\leq \Gamma}$ that is equal to $a^{H \setminus \Gamma}$ on $X^{H \setminus \Gamma}$. Then $a^{\leq \Gamma}$ commutes with the shift action s on $X^{\leq \Gamma}$ and since $(\gamma^s)_*(\gamma^{a^{H \setminus \Gamma}})_*(\mu_H)^{H \setminus \Gamma} = \mu_{\gamma H \gamma^{-1}}^{(\gamma H \gamma^{-1}) \setminus \Gamma}$ it follows from invariance of θ that the action $\gamma^b = \gamma^s \gamma^{a^{\leq \Gamma}}$ preserves the measure ν . We let $\mathbf{b} = \Gamma \curvearrowright^b (Y, \nu)$. Then θ is a factor of \mathbf{b} via the map $f \mapsto H_f$. Let (Z, η) be a standard non-atomic probability space, and let $\mathbf{b} \otimes_{\theta} s_{\theta, \eta}$ denote the relatively independent joining of \mathbf{b} and $s_{\theta, \eta}$ over θ .

We now apply Lemma 7.1 to $s_{\theta, \eta}$. Given $F \subseteq \Gamma$ finite and $\epsilon > 0$ there exists a measurable $J : Z^{\leq \Gamma} \rightarrow \text{COS}(\Gamma)$ such that $\eta^{\theta \setminus \Gamma}(Z_0) \geq 1 - \epsilon$ where $Z_0 = \{g \in Z^{\leq \Gamma} : J(\gamma^s \cdot g) = \gamma^\ell \cdot J(g) \text{ for all } \gamma \in F\}$, and with $J(g) \in \Gamma_g \setminus \Gamma = H_g \setminus \Gamma$ for all $g \in Z^{\leq \Gamma}$. We let $\varphi : Y \times Z^{\leq \Gamma} \rightarrow X$ be the map defined $(\nu \otimes_{\theta} \eta^{\leq \Gamma})$ -almost everywhere by $\varphi(f, g) = f(J(g))$. Then for all $g \in Z_0$ and $\gamma \in F$ we have

$$\varphi(\gamma^{b \times s}(f, g)) = \gamma^a((\gamma^s f)(J(\gamma^s g))) = \gamma^a(f(J(g))) = \gamma^a \varphi(f, g))$$

and

$$\begin{aligned} \varphi_*(\nu \otimes_{\theta} \eta^{\leq \Gamma}) &= \int_H \int_g \int_f \delta_{f(J(g))} d\mu_H^{H \setminus \Gamma} d\eta^{H \setminus \Gamma} d\theta \\ &= \int_H \sum_{t \in H \setminus \Gamma} \int_{\{g: J(g)=t\}} \mu_H d\eta^{H \setminus \Gamma} d\theta = \int_H \mu_H d\theta = \mu. \end{aligned}$$

It then follows that $\mathbf{a} < \mathbf{b} \otimes_{\theta} s_{\theta}$ since for any measurable partition $A_0, \dots, A_{k-1} \subseteq X$ of X , the sets $B_0 = \varphi^{-1}(A_0), \dots, B_{k-1} = \varphi^{-1}(A_{k-1})$ form a measurable partition of $Y \times X^{\leq \Gamma}$ satisfying $|\mu(\gamma^a A_i \cap A_j) - (\nu \otimes_{\theta} \eta^{\leq \Gamma})(\gamma^{b \times s} B_i \cap B_j)| < \epsilon$ for all $\gamma \in F$. By the Rokhlin skew-product theorem there exist a standard probability space (Z_1, η_1) and an isomorphism Ψ of \mathbf{a} with a skew product action $\mathbf{d} = (Z_1, \eta_1) \times \theta$ on the space $(Z_1 \times \text{Sub}(H), \eta_1 \times \theta)$. The isomorphism Ψ is of the form $\Psi(x) = (\Psi_0(x), \Gamma_x)$ and so the restriction Ψ_H of Ψ_0 to $X_H = \{x : \Gamma_x = H\}$ is an isomorphism of (X_H, μ_H) with (Z_1, η_1) almost surely. We now define an isomorphism $\Phi : Y \rightarrow Z_1^{\leq \Gamma}$ of \mathbf{b} with s_{θ, η_1} by taking $H_{\Phi(f)} = H_f$ and $\Phi(f)(H\gamma) = \Psi_{\gamma^{-1}H\gamma}((\gamma^{-1})^a(f(H\gamma)))$, where $H = H_f$. This is almost everywhere well defined

since $f(H\gamma) \in X_H$ almost surely, which ensures that $(\gamma^{-1})^a(f(H\gamma))$ is independent of our choice of representative for the coset $H\gamma$, and $(\gamma^{-1})^a(f(H\gamma)) \in X_{\gamma^{-1}H\gamma}$ so that we may apply $\Psi_{\gamma^{-1}H\gamma}$. The map Φ is equivariant since if $H_f = H$ then $H_{\delta^b f} = \delta H \delta^{-1}$ and $\Phi(\delta^b f)(\delta H \delta^{-1} \gamma) = \Psi_{\gamma^{-1} \delta H \delta^{-1} \gamma}((\gamma^{-1})^a(\delta^b f(\delta H \delta^{-1} \gamma))) = \Psi_{\gamma^{-1} \delta H (\gamma^{-1} \delta)^{-1}}((\gamma^{-1} \delta)^a(f(H \delta^{-1} \gamma))) = \Phi(f)(H \delta^{-1} \gamma) = (\delta^s \Phi(f))(\delta H \delta^{-1} \gamma)$. Finally, $\Phi_* \nu = \eta_1^{\theta \setminus \Gamma}$ since

$$\begin{aligned} \Phi_* \nu &= \int_H \Phi_* \mu_H^{H \setminus \Gamma} d\theta = \int_H \prod_{H\gamma \in H \setminus \Gamma} (\Psi_{\gamma^{-1}H\gamma})_* (\gamma^{-1})_*^a \mu_H d\theta \\ &= \int_H \prod_{H\gamma \in H \setminus \Gamma} (\Psi_{\gamma^{-1}H\gamma})_* \mu_{\gamma^{-1}H\gamma} d\theta = \int_H \eta_1^{H \setminus \Gamma} d\theta = \eta_1^{\theta \setminus \Gamma} \end{aligned}$$

and so $\mathbf{b} \cong s_{\theta, \eta_1}$. Since $H_f = H_{\Phi(f)}$, this extends to an isomorphism of $\mathbf{b} \otimes_{\theta} s_{\theta, \mu}$ with $s_{\theta, \eta_1} \otimes_{\theta} s_{\theta, \eta} \cong s_{\theta, \eta_1 \times \eta} \cong s_{\theta, \eta}$, as was to be shown.

We next show that $(*)$ holds under the assumption that θ is ergodic. Let $i \in \mathbb{N} \cup \{\infty\}$ be the index of θ . If i is finite then the orbit of almost every $H \in \text{Sub}(\Gamma)$ is finite so by ergodicity of θ there exists $H_0 \leq \Gamma$ of index i such that θ concentrates on the conjugates of H_0 . Then for some spaces (Z_1, η_1) and (Z_2, η_2) we have $\mathbf{a} \cong \iota_{\eta_1} \times \mathbf{a}_{\Gamma/H_0}$ and $\mathbf{d} \cong \iota_{\eta_2} \times \mathbf{a}_{\Gamma/H_0}$, where \mathbf{a}_{Γ/H_0} denotes the action of Γ on the left cosets of H_0 with normalized counting measure. Thus $\mathbf{a} \sim_s \mathbf{d}$. If $i = \infty$ then we let $\mathbf{a} = \int_Z \mathbf{a}_z d\eta$ and $\mathbf{d} = \int_W \mathbf{d}_w d\rho$ be the ergodic decompositions of \mathbf{a} and \mathbf{d} respectively. By Proposition 3.9, $\text{type}(\mathbf{a}_z) = \theta$ and $\text{type}(\mathbf{d}_w) = \theta$ almost surely, and \mathbf{a}_z and \mathbf{d}_w are non-atomic almost surely since θ is infinite index. Letting \mathbf{b} be any non-atomic ergodic action of type θ the above case implies that $\mathbf{a} \sim_s \mathbf{b} \sim_s \mathbf{d}$.

Finally, we show that $(*)$ holds in general. Let $\theta = \int_{w \in W} \theta_w d\rho$ be the ergodic decomposition of θ . We then obtain corresponding decompositions $\mathbf{a} = \int_w \mathbf{a}_w d\rho$ and $\mathbf{d} = \int_w \mathbf{d}_w d\rho$ of \mathbf{a} and \mathbf{d} with $\text{type}(\mathbf{a}_w) = \theta_w = \text{type}(\mathbf{d}_w)$ almost surely. The above cases imply that $\mathbf{a}_w \sim_s \mathbf{d}_w$ almost surely. Theorem 3.13 then implies $\mathbf{a} \sim_s \mathbf{d}$. \square

Proof of Theorem 1.8(2). Let $\theta = \text{type}(\mathbf{a}) = \text{type}(\mathbf{b})$. If θ is ergodic then by Proposition 3.9 almost every ergodic component of \mathbf{a} and almost every ergodic component of \mathbf{b} have type θ and so Theorem 1.8(1) and Corollary 4.4 imply that $\mathbf{a} \sim \iota_{\eta_1} \times \mathbf{d}$ and $\mathbf{b} \sim \iota_{\eta_2} \times \mathbf{d}$ for some ergodic \mathbf{d} of type θ and some spaces (Z_1, η_1) , (Z_2, η_2) . Since Γ is amenable, \mathbf{d} is not strongly ergodic, and since θ is infinite index, \mathbf{d} is non-atomic, so by [AW13, Theorem 3] $\mathbf{d} \sim \iota \times \mathbf{d}$ and thus $\mathbf{a} \sim \mathbf{b}$. The general case now follows by considering the ergodic decomposition of θ . \square

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A. Appendix. Ultraproducts of measure preserving actions

In this appendix we establish some properties of ultraproducts of measure spaces and actions.

Notation. We refer to [CKT-D13] for background on ultraproducts of measure preserving actions and also [ES08] for background on ultraproducts of measure spaces. Our notation has some changes from that of [CKT-D13] and is as follows. Given a sequence $\mathbf{a}_n = \Gamma \curvearrowright^{a_n} (X_n, \mu_n)$, $n \in \mathbb{N}$, of measure preserving actions of Γ and a non-principal ultrafilter \mathcal{U} on \mathbb{N} , we denote by $(\prod_n \mathbf{a}_n)/\mathcal{U} = \Gamma \curvearrowright^{(\prod_n a_n)\mathcal{U}} ((\prod_n X_n)/\mathcal{U}, (\prod_n \mu_n)/\mathcal{U})$, or simply $\mathbf{a}_{\mathcal{U}} = \Gamma \curvearrowright^{a_{\mathcal{U}}} (X_{\mathcal{U}}, \mu_{\mathcal{U}})$ when there is no danger of confusion, the corresponding ultraproduct of the sequence (\mathbf{a}_n) . We let $[x_n]$ denote the equivalence class of the sequence $(x_n) \in \prod_n X_n$ in $X_{\mathcal{U}}$ and we let $[B_n]$ denote the subset of $X_{\mathcal{U}}$ determined by the sequence $(B_n) \in \prod_n \mathcal{B}(X_n)$ of Borel sets. When $x_n = x$ for all n , we write $[x]$ for $[x_n]$, and when $B_n = B$ for all n , we write $[B]$ for $[B_n]$. Then $\mathbf{A}_{\mathcal{U}} = \mathbf{A}_{\mathcal{U}}(X_{\mathcal{U}}) = \{[B_n] : (B_n) \in \prod_n \mathcal{B}(X_n)\}$ is an algebra of subsets of $X_{\mathcal{U}}$ and $\mu_{\mathcal{U}}$ is the unique measure on the σ -algebra $\mathcal{B}_{\mathcal{U}}(X_{\mathcal{U}}) = \sigma(\mathbf{A}_{\mathcal{U}})$ whose value on $[A_n] \in \mathbf{A}_{\mathcal{U}}$ is $\mu_{\mathcal{U}}([A_n]) = \lim_{n \rightarrow \mathcal{U}} \mu_n(A_n)$. We note that every element of $\mathcal{B}_{\mathcal{U}}$ is within a $\mu_{\mathcal{U}}$ -null set of an element of $\mathbf{A}_{\mathcal{U}}$.

The following proposition deals with lifting measure disintegrations to ultraproducts.

PROPOSITION A.1. *Suppose that for each $n \in \mathbb{N}$ the Borel map $\pi_n : (Y_n, \nu_n) \rightarrow (Z_n, \eta_n)$ factors $\mathbf{b}_n = \Gamma \curvearrowright^{b_n} (Y_n, \nu_n)$ onto $\mathbf{d}_n \curvearrowright^d (Z_n, \eta_n)$, and let $\nu_n = \int_{z \in Z_n} \nu_z^n d\eta_n(z)$ be the disintegration of ν_n over η_n with respect to π_n . Let $\mathbf{b}_{\mathcal{U}} = \Gamma \curvearrowright^{b_{\mathcal{U}}} (Y_{\mathcal{U}}, \nu_{\mathcal{U}})$ and $\mathbf{d}_{\mathcal{U}} = \Gamma \curvearrowright^{d_{\mathcal{U}}} (Z_{\mathcal{U}}, \eta_{\mathcal{U}})$ be the ultraproducts of the sequences (\mathbf{b}_n) and (\mathbf{d}_n) respectively. Then the map $\pi_{\mathcal{U}} : Y_{\mathcal{U}} \rightarrow Z_{\mathcal{U}}$ given by $\pi_{\mathcal{U}}([y_n]) = [\pi_n(y_n)]$ factors $\mathbf{b}_{\mathcal{U}}$ onto $\mathbf{d}_{\mathcal{U}}$. If for $[z_n] \in Z_{\mathcal{U}}$ we let $\nu_{[z_n]} = (\prod_n \nu_{z_n}^n)/\mathcal{U}$ then:*

- (I) *each of the measures $\nu_{[z_n]}$ is a probability measure on $(Y_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}(Y_{\mathcal{U}}))$ and almost surely $\nu_{[z_n]}$ concentrates on $\pi_{\mathcal{U}}^{-1}([z_n])$;*
- (II) *for each $D \in \mathcal{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ the map $(Z_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}(Z_{\mathcal{U}})) \rightarrow ([0, 1], \mathcal{B}([0, 1]))$ sending $[z_n] \mapsto \nu_{[z_n]}(D)$ is measurable and $\nu_{\mathcal{U}}(D) = \int_{[z_n] \in Z_{\mathcal{U}}} \nu_{[z_n]}(D) d\eta_{\mathcal{U}}([z_n])$;*
- (III) *if $[z_n] \mapsto \mu_{[z_n]}$ is another assignment satisfying (I) and (II) then for all $D \in \mathcal{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ almost surely $\mu_{[z_n]}(D) = \nu_{[z_n]}(D)$.*

Additionally, for almost all $[z_n] \in Z_{\mathcal{U}}$ and every $\gamma \in \Gamma$ we have $(\gamma^{b_{\mathcal{U}}})_ \nu_{[z_n]} = \nu_{\gamma^{d_{\mathcal{U}}}[z_n]}$.*

Proof. It is clear that $\pi_{\mathcal{U}}$ factors $\mathbf{b}_{\mathcal{U}}$ onto $\mathbf{d}_{\mathcal{U}}$. Property (I) follows from the fact that for each n and $z \in Z_n$, each ν_z^n is a Borel probability measure on Y_n and almost surely ν_z^n concentrates on $\pi_n^{-1}(\{z\})$. Now let \mathbf{D} be the collection of all subsets of $Y_{\mathcal{U}}$ satisfying (II). Given $[A_n] \in \mathbf{A}_{\mathcal{U}}$ and $V \subseteq [0, 1]$ open we have $\nu_{[z_n]}([A_n]) \in V$ if and only if $[z_n] \in \{[z] : \nu_z^n(A_n) \in V\}$, so that $[z_n] \mapsto \nu_{[z_n]}([A_n])$ is measurable. As in [ES08, Lemma 2.2] we have

$$\begin{aligned} \int_{[z_n]} \nu_{[z_n]}(A_n) d\eta_{\mathcal{U}} &= \int_{[z_n]} \lim_{n \rightarrow \mathcal{U}} \nu_z^n(A_n) d\eta_{\mathcal{U}} \\ &= \lim_{n \rightarrow \mathcal{U}} \int_{z \in Z_n} \nu_z^n(A_n) d\eta_n = \lim_{n \rightarrow \mathcal{U}} \nu_n(A_n) = \nu_{\mathcal{U}}([A_n]), \end{aligned}$$

which shows that $[A_n] \in \mathbf{D}$. Thus $\mathbf{A}_{\mathcal{U}} \subseteq \mathbf{D}$, and it is clear that \mathbf{D} is a monotone class so $\mathcal{B}_{\mathcal{U}} \subseteq \mathbf{D}$, which shows (II). Suppose now that $[z_n] \mapsto \mu_{[z_n]}$ satisfies (I) and (II). Then for each $[B_n] \in \mathbf{A}_{\mathcal{U}}(Z_{\mathcal{U}})$ and $D \in \mathcal{B}_{\mathcal{U}}(Y_{\mathcal{U}})$ we have

$$\int_{[B_n]} \mu_{[z_n]}(D) d\eta_{\mathcal{U}} = \nu_{\mathcal{U}}(D \cap \pi_{\mathcal{U}}^{-1}([B_n])) = \int_{[B_n]} \nu_{[z_n]}(D) d\eta_{\mathcal{U}}$$

so that $\mu_{[z_n]}(D) = \nu_{[z_n]}(D)$ almost surely, so that (III) holds.

For the last statement let $B_n \subseteq Z_n$ be an invariant η_n -conull set on which $(\gamma^{b_n})_* v_z^n = v_{\gamma^{d_n} z}^n$ for all $\gamma \in \Gamma$. Then for all $[z_n]$ in the $\eta_{\mathcal{U}}$ -conull set $[B_n] \subseteq Z_{\mathcal{U}}$ we have for all $\gamma \in \Gamma$ and $[A_n] \in \mathcal{A}_{\mathcal{U}}(Y_{\mathcal{U}})$ that

$$(\gamma^{d_{\mathcal{U}}})_* v_{[z_n]}(A_n) = \lim_{n \rightarrow \mathcal{U}} (\gamma^{d_n})_* v_{z_n}^n(A_n) = \lim_{n \rightarrow \mathcal{U}} v_{\gamma^{d_n} z}^n(A_n) = v_{\gamma^{d_{\mathcal{U}}} [z_n]}([A_n])$$

so that $(\gamma^{d_{\mathcal{U}}})_* v_{[z_n]} = v_{\gamma^{d_{\mathcal{U}}} [z_n]}$. \square

The next proposition describes the ultrapower of a standard probability space with atoms.

PROPOSITION A.2. *Let (Z, η) be a standard probability space and let $A \subseteq Z$ be the set of atoms of (Z, η) .*

- (1) *If (Z, η) is discrete then $(\text{MALG}_{\eta}, d_{\eta})$ is a compact metric space homeomorphic to 2^A with the product topology, and the map $I_{\mathcal{U}} : \text{MALG}_{\eta_{\mathcal{U}}} \rightarrow \text{MALG}_{\eta}$ given by $I_{\mathcal{U}}([B_n]) = \lim_{n \rightarrow \mathcal{U}} B_n = \{z \in A : \{n : z \in B_n\} \in \mathcal{U}\}$ is a measure algebra isomorphism.*
- (2) *In general, $[A] = \{[z] : z \in A\} \subseteq Z_{\mathcal{U}}$ is the set of all atoms of $\eta_{\mathcal{U}}$ and the restriction $\eta|A$ of η to A is isomorphic as a measure space to the restriction $\eta_{\mathcal{U}}|[A]$ of $\eta_{\mathcal{U}}$ to $[A]$ via the map $z \mapsto [z]$. Under this isomorphism, letting $C = Z \setminus A$, we may identify $(Z_{\mathcal{U}}, \eta_{\mathcal{U}})$ with $([C] \sqcup A, (\eta|C)_{\mathcal{U}} + \eta|A)$.*

Proof. First suppose that (Z, η) is discrete. Without loss of generality we may assume that $Z = A$. As sets we may identify MALG_{η} with 2^A . Let B_0, B_1, \dots be a sequence in 2^A converging in the product topology to some set $B \in 2^A$. Given $\epsilon > 0$, let $F \subseteq A$ be a finite set such that $\eta(A \setminus F) < \epsilon$. For all large enough n , B_n and B agree on F , so that $\eta(B_n \Delta B) < \eta(A \setminus F) < \epsilon$ and thus $d_{\eta}(B_n, B) \rightarrow 0$. This shows that the map $2^A \rightarrow \text{MALG}_{\eta}$ is a continuous bijection from the compact Hausdorff space 2^A (with the product topology) to $(\text{MALG}_{\eta}, d_{\eta})$, so it is a homeomorphism. It is clear that the map φ taking $B \subseteq A$ to $[B] \subseteq [A]$ is an isometric embedding of MALG_{η} to $\text{MALG}_{\eta_{\mathcal{U}}}$ that preserves all Boolean operations. If now $[B_n] \subseteq [A]$ and $\lim_{n \rightarrow \mathcal{U}} B_n = B$ then $d_{\eta_{\mathcal{U}}}([B_n], [B]) = \lim_{n \rightarrow \mathcal{U}} d_{\eta}(B_n, B) = 0$ so that $[B_n] = [B]$ and thus $\varphi^{-1} = I_{\mathcal{U}}$, which completes the proof of (1). Part (2) follows since $(Z_{\mathcal{U}}, \eta_{\mathcal{U}})$ decomposes as $([C] \sqcup [A], (\eta|C)_{\mathcal{U}} + (\eta|A)_{\mathcal{U}})$ and part (1) shows that $([A], (\eta|A)_{\mathcal{U}}) \cong (A, \eta)$. \square

THEOREM A.3. *Let $\mathbf{a}_0, \mathbf{a}_1, \dots$ be a sequence of measure preserving actions of Γ on the standard probability space (X, μ) and let $\mathbf{a}_{\mathcal{U}} = \Gamma \curvearrowright^{\mathbf{a}_{\mathcal{U}}} (X_{\mathcal{U}}, \mu_{\mathcal{U}})$ be their ultrapower. Let $\mathbf{M}_0 \subseteq \text{MALG}_{\mu_{\mathcal{U}}}$ be any subset such that $(\mathbf{M}_0, d_{\mu_{\mathcal{U}}}|\mathbf{M}_0)$ is separable. Then there exists an invariant measure sub-algebra \mathbf{M} of $\text{MALG}_{\mu_{\mathcal{U}}}$ containing \mathbf{M}_0 that is isomorphic as a measure algebra to MALG_{μ} .*

Proof. Let $A \subseteq X$ be the collection of atoms of X and let $C = X \setminus A$. By Proposition A.2(2), $[A] \subseteq X$ is the discrete part of $\mu_{\mathcal{U}}$ and $x \mapsto [x]$ is an isomorphism $\mu|A \cong \mu_{\mathcal{U}}|[A]$. Define a function $S_{\mathcal{U}} : \text{MALG}_{\mu_{\mathcal{U}}} \rightarrow \text{MALG}_{\mu_{\mathcal{U}}}$ first on subsets $D \subseteq [C]$ by taking $S_{\mathcal{U}}(D)$ to be any subset of D satisfying $\mu_{\mathcal{U}}(S_{\mathcal{U}}(D)) = \frac{1}{2}\mu_{\mathcal{U}}(D)$, and then extending this to all of $\text{MALG}_{\mu_{\mathcal{U}}}$ by taking $S_{\mathcal{U}}(D) = S_{\mathcal{U}}(D \cap [C]) \sqcup (D \cap [A])$. Fix a countable dense subset \mathbf{M}_1 of \mathbf{M}_0 and let $\mathbf{B}_0 \subseteq \text{MALG}_{\mu_{\mathcal{U}}}$ be a countable Boolean algebra

containing $\mathbf{M}_1 \cup \{[x] : x \in A\}$ and closed under the functions $S_{\mathcal{U}}$ and $\gamma^{a_{\mathcal{U}}}$ for all $\gamma \in \Gamma$. Then the σ -algebra $\mathbf{M} = \sigma(\mathbf{B}_0)$ equipped with $\mu_{\mathcal{U}}$ is an invariant countably generated measure sub-algebra of $\text{MALG}_{\mu_{\mathcal{U}}}$ containing \mathbf{M}_0 . Since \mathbf{B}_0 is closed under $S_{\mathcal{U}}$, the atoms of \mathbf{B}_0 , and hence also those of \mathbf{M} , must be contained in $[A]$, and as \mathbf{M} contains $\{[B] : B \subseteq A\}$, the discrete part of \mathbf{M} is isomorphic to the discrete part of MALG_{μ} . It follows that $\mathbf{M} \cong \text{MALG}_{\mu}$. \square

PROPOSITION A.4. *Let $\mathbf{a} = \Gamma \curvearrowright^a (X, \mu)$ and $\mathbf{b} = \Gamma \curvearrowright^b (Y, \nu)$ be measure preserving actions of Γ . If \mathbf{a} is weakly contained in \mathbf{b} then the measure space (X, μ) is a quotient of the measure space (Y, ν) . If \mathbf{a} and \mathbf{b} are weakly equivalent then (X, μ) is isomorphic to (Y, ν) . In particular, the identity actions \mathbf{t}_{η_1} and \mathbf{t}_{η_2} are weakly equivalent if and only if (Z_1, η_1) and (Z_2, η_2) are isomorphic measure spaces.*

Proof. Suppose first that $\mathbf{a} < \mathbf{b}$. Let $\phi : X \rightarrow K = 2^{\mathbb{N}}$ be any Borel isomorphism and let $\lambda = (\Phi^{\phi, a})_* \mu$. Then $\mathbf{a} \cong \Gamma \curvearrowright^s (K^{\Gamma}, \lambda)$ and as $\mathbf{a} < \mathbf{b}$ there exists $\lambda_n = (\Phi^{\phi_n, b})_* \nu \in E(\mathbf{b}, K)$ with $\lambda_n \rightarrow \lambda$. By Proposition 3.11 $\Gamma \curvearrowright^s (K^{\Gamma}, \lambda)$ is a factor of the ultrapower $\mathbf{b}_{\mathcal{U}}$ of \mathbf{b} via $\Phi^{\phi, b_{\mathcal{U}}}$, where ϕ is the ultralimit of the ϕ_n . Thus \mathbf{a} is also a factor of $\mathbf{b}_{\mathcal{U}}$ so by Theorem A.3 this implies (X, μ) is a factor of (Y, ν) .

Now suppose that \mathbf{a} and \mathbf{b} are weakly equivalent. Then the measure spaces (X, μ) and (Y, ν) are factors of each other, say $\pi : (Y, \nu) \rightarrow (X, \mu)$ and $\varphi : (X, \mu) \rightarrow (Y, \nu)$. Let $A \subseteq X$ be the set of atoms of X and let $B \subseteq Y$ be the set of atoms of Y . If $\mu(A) = 0$ then we are done since this implies both (X, μ) and (Y, ν) are non-atomic. So suppose that $\mu(A) > 0$. It is clear that $A \subseteq \varphi^{-1}(B)$ and $B \subseteq \pi^{-1}(A)$, hence $\mu(A) = \nu(B)$. Additionally, $\mu(\varphi^{-1}(B) \setminus A) = 0$, otherwise $\nu(B) = \mu(\varphi^{-1}(B)) > \mu(A)$. Similarly, $\nu(\pi^{-1}(A) \setminus B) = 0$. Thus $\varphi^{-1} : (\text{MALG}_{\nu_B}, d_{\nu_B}) \rightarrow (\text{MALG}_{\mu_A}, d_{\mu_A})$ and $\pi^{-1} : (\text{MALG}_{\mu_A}, d_{\mu_A}) \rightarrow (\text{MALG}_{\nu_B}, d_{\nu_B})$ are isometric embeddings of compact metric spaces (Proposition A.2), so it follows that both π^{-1} and φ^{-1} are in fact isometric isomorphisms. Since these maps are also Boolean algebra homomorphisms it follows that both are measure algebra isomorphisms. This shows that the discrete parts of (X, μ) and (Y, ν) are isomorphic, from which it follows that (X, μ) and (Y, ν) are isomorphic. \square

B. Appendix. Stable weak containment

In this appendix we establish some basic properties of stable weak containment of measure preserving actions. Our development mirrors our development of weak containment of measure preserving actions.

Definition B.1. Let \mathcal{A} and \mathcal{B} be two sets of measure preserving actions of Γ . We say that \mathcal{A} is *stably weakly contained* in \mathcal{B} , written $\mathcal{A} \prec_s \mathcal{B}$, if for every $\Gamma \curvearrowright^a (X, \mu) = \mathbf{a} \in \mathcal{A}$, for any Borel partition A_0, \dots, A_{k-1} of X , $F \subseteq \Gamma$ finite, and $\epsilon > 0$, there exist non-negative reals $\alpha_0, \dots, \alpha_{m-1}$ with $\sum_{i < m} \alpha_i = 1$ along with actions $\Gamma \curvearrowright^{b_i} (Y_i, \nu_i) = \mathbf{b}_i \in \mathcal{B}$, $i < m$, and a Borel partition B_0, \dots, B_{k-1} of $\sum_{i < m} Y_i$ such that

$$|\mu(\gamma^a A_i \cap A_j) - \left(\sum_{i < m} \alpha_i \tilde{\nu}_i \right) (\gamma^{\sum_{i < m} b_i} B_i \cap B_j)| < \epsilon$$

for all $i, j < k$ and $\gamma \in F$. (See §3.2 for notation.)

The relation \prec_s is a reflexive and transitive relation on sets of measure preserving actions. We call \mathcal{A} and \mathcal{B} *stably weakly equivalent*, written $\mathcal{A} \sim_s \mathcal{B}$, if both $\mathcal{A} \prec_s \mathcal{B}$ and $\mathcal{B} \prec_s \mathcal{A}$. We write $\mathbf{a} \prec_s \mathcal{B}$, $\mathcal{A} \prec_s \mathbf{b}$, and $\mathbf{a} \prec_s \mathbf{b}$ for $\{\mathbf{a}\} \prec_s \mathcal{B}$, $\mathcal{A} \prec_s \{\mathbf{b}\}$, and $\{\mathbf{a}\} \prec_s \{\mathbf{b}\}$ respectively, and similarly with \sim_s in place of \prec_s .

It is clear that $\mathbf{a} \prec_s \mathbf{b}$ if and only if $\mathbf{a} \prec \{\iota_{\eta_\alpha} \times \mathbf{b} : \alpha = (\alpha_0, \dots, \alpha_{m-1}) \in [0, 1]^m, \sum_{i < m} \alpha_i = 1, m \in \mathbb{N}\}$, so by Lemma 3.8 we have $\mathbf{a} \prec_s \mathbf{b}$ if and only if $\mathbf{a} \prec \iota \times \mathbf{b}$ if and only if $\iota \times \mathbf{a} \prec \iota \times \mathbf{b}$. From this point of view, Theorem 1.2 says that if \mathbf{a} is ergodic then $\mathbf{a} \prec_s \mathbf{b}$ if and only if $\mathbf{a} \prec \mathbf{b}$. Theorem 1.1 implies that $\mathbf{a} \prec_s \mathbf{b}$ if and only if $E(\mathbf{a}, K) \subseteq \overline{\text{co}}E(\mathbf{b}, K)$ for every compact Polish space K , and $\mathbf{a} \sim_s \mathbf{b}$ if and only if $\overline{\text{co}}E(\mathbf{a}, K) = \overline{\text{co}}E(\mathbf{b}, K)$ for every compact Polish space K . More generally, we have the following analogue of Proposition 3.6, which can be proved directly by using the same methods.

PROPOSITION B.2. *Let \mathcal{A} and \mathcal{B} be sets of measure preserving actions of Γ . Then the following are equivalent:*

- (1) $\mathcal{A} \prec_s \mathcal{B}$;
- (2) $\bigcup_{\mathbf{d} \in \mathcal{A}} E(\mathbf{d}, K) \subseteq \overline{\text{co}}(\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, K))$ for every finite K ;
- (3) $\bigcup_{\mathbf{d} \in \mathcal{A}} E(\mathbf{d}, K) \subseteq \overline{\text{co}}(\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, K))$ for every compact Polish K ;
- (4) $\bigcup_{\mathbf{d} \in \mathcal{A}} E(\mathbf{d}, 2^{\mathbb{N}}) \subseteq \overline{\text{co}}(\bigcup_{\mathbf{b} \in \mathcal{B}} E(\mathbf{b}, 2^{\mathbb{N}}))$.

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